63-2-4

NOTES ON THE THEORY OF ECONOMIC PLANNING

BY

ROY RADNER

TECHNICAL REPORT NO. 9
JANUARY 1963

PREPARED UNDER CONTRACT Nonr-222 (77)
(NR-047-029)
FOR
OFFICE OF NAVAL RESEARCH

This research was supported in part by the Office of Naval Research under Contract ONR 222 (77) with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

CENTER FOR RESEARCH IN MANAGEMENT SCIENCE
UNIVERSITY OF CALIFORNIA
Berkeley 4, California

NOTES ON THE THEORY OF ECONOMIC PLANNING

BY

ROY RADNER

TECHNICAL REPORT NO. 9
JANUARY 1963

PREPARED UNDER CONTRACT Nonr-222 (77)
(NR-047-029)
FOR
OFFICE OF NAVAL RESEARCH

This research was supported in part by the Office of Naval Research under Contract ONR 222 (77) with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

CENTER FOR RESEARCH IN MANAGEMENT SCIENCE
UNIVERSITY OF CALIFORNIA
Berkeley 4, California

ACKNOWLEDGMENTS

This report has been prepared in connection with the research project "Economics of Information and Organization," supported by the Office of Naval Research under Contract Nonr-222(77) (NR-047-029) with the University of California, Berkeley. Reproduction of these notes in whole or in part is permitted for any purpose of the United States Government. Most of the work was done while I was a Fellow of the Guggenheim Foundation, whose support I also gratefully acknowledge.

I thank the Center of Economic Research, Athens, Greece, and its Director, Professor A. G. Papandreou, for giving me the opportunity of presenting this material in a series of lectures during the summer of 1962.

Finally, Professor C. B. McGuire helped me a great deal with his comments on the manuscript.

TABLE OF CONTENTS

CHAPTER V. Proportional Growth Programs

Section 1. Introduction

2. Fastest Growing Proportional Growth Without Consumption

3. Interest Rates for Efficient Proportional Growth Programs with Consumption

4. Do Optimal Programs Tend Towards Proportional Growth in the Long Run?

INTRODUCTION

The aim of these notes is to introduce the reader to some mathematical models of economic planning on a national scale, and to a number of theoretical results on the properties of optimal economic programs.

In addition to describing a fairly general framework for the mathematical analysis of economic planning, I describe a number of special models (Chapters I and II). These models have in common the following features:

(1) They either have been used in applications, or appear to have promise of applicability.

(2) Planning is formulated in terms of real goods and services, or index numbers of real quantities, rather than in terms of financial magnitudes.

(3) The models can, in principle, be used with a relatively high degree of disaggregation by commodities.

The theoretical results presented are of two types. First, I review some of the literature concerning the properties of optimal paths of economic growth. In this literature, an important topic is the role of shadow prices and interest rates as indicators of optimality (Chapters III and V). Much attention has also been given to proportional (balanced) growth, and the tendency of optimal programs to approximate proportional growth (Chapter V).

The second group of results, which have not been previously published, concern a rather special model — special with regard to both the description of production possibilities and the criterion of optimality. For this model I discuss in some detail the properties of optimal programs (Chapter IV). For both finite and infinite planning horizons, I give formulas for optimal time sequences of consumption, investment, and allocation of resources. The long-run growth rates, directions of growth, and shadow interest rates are also given. Using these results one can study the way in which the optimal path depends upon the various parameters of the technology and of the preference criterion. In particular, one can get some interesting results about the influence of time preference.

The results for this special case illustrate the various general theorems mentioned above. In addition, the special case is general enough, and the computations required to determine the optimal programs are simple enough, to make the model appear attractive for applied work.

I have tried to present the various theorems in a fairly precise fashion, and therefore have adopted a mathematical presentation. On the other hand, I have included proofs in only a few of the simplest cases. This limitation was a consequence of the time limits of the lectures for which these notes were written, and of the interests of the audience.[+] Readers who are interested in proofs can follow up the references to the literature, except in the case of Chapter IV. In that chapter I have given some indications of the method of solution, since I use the technique of "dynamic programming", and this technique is relatively new to the theory of economic planning. (A paper giving complete proofs of the results in Chapter IV will be available soon.)

Limitations on the Scope of the Theory Presented

The literature on the theory of economic planning, though primitive in many respects, still covers a wide field of topics, of which only a few are included in these notes. I should try to make clear at the beginning the limitations that have been imposed, again by lack of time, and also by the limits of my own competence.

(1) The theories presented are intended to apply primarily to planning on a national scale.

(2) The planning considered is "technological" in the sense that the planning takes place within technological, but not behavioristic or financial, constraints. Thus, I do not explicitly consider models of autonomous determination of the behavior of

---

[+]An elementary knowledge of the differential calculus and matrix algebra should enable the reader to follow these notes.

economic agents such as consumers and investors. It is difficult, of course, to make a sharp distinction between technological and behavioristic determinants in the economy; for example, whether one treats the consumption of food as a planned input into the activity of producing labor, or as determined by a demand function for food, will depend upon institutional features of the particular problem being considered. Even in a free market economy, however, there is some interest in comparing the hypothetical results of a technologically planned program with the historical or projected development of the economy.

(3) The models considered are aggregate in terms of individuals (consumers, firms). Planning is discussed in terms of total consumption or total production of the various commodities.

(4) There is no discussion of techniques for decentralizing the planning process or the process of carrying out the plan. (However, one result of Chapter III, Section 1, bears on this point, and I also give some references to the literature on the subject.)

(5) There is no discussion of uncertainty. Indeed, there has been practically no theoretical investigation of uncertainty in economic planning.[+]

Plan of the Notes

A necessary step in the mathematical analysis of a planning problem is, of course, a precise formulation of the problem. In the approach that I have followed, the specification of the problem can be divided into two parts, a specification of the set of programs that are technologically feasible, and a specification of the criterion to be used in comparing alternative programs. These two tasks are interrelated in so far as the type of criterion used is limited by the terms in which one describes the programs. For example, if consumption is described only in terms of total consumption of each commodity, then one cannot compare

---

[+]See, however, J. Mirlees, "The influence of uncertainty on the optimum rate of investment," Ph.D. Dissertation, Cambridge University, Cambridge, England, 1962.

programs on the basis of the distribution of consumption among individual consumers. Chapter I presents a general framework for the description of technological possibilities, in a dynamic context, together with a number of special cases, including linear activity analysis, the dynamic input-output model, and certain special production functions. Chapter II discusses an array of alternative criteria for comparing programs. In particular, some attention is given to the problem of defining criteria for programs with an infinite horizon.

Certain criteria are of interest, not because they directly express value judgments about an economic program, but because it is hoped that their use will lead to the selection of programs that are preferred in some more basic sense. I have in mind here such criteria as present value, rate of return, and the benefit-cost ratio. In Chapter III, under the heading "Derived Criteria", I discuss the rationale, or lack of rationale, for the use of these criteria.

In Chapter IV I fit together various elements introduced in Chapters I and II in the form of a complete, but special, model, and I discuss in some detail the calculation and properties of optimal programs for this special case.

Much of the recent literature on the theory of optimal economic growth deals with proportional growth, and in particular with the following two questions: (1) What is the relation between the rate of growth and the shadow rate of interest in an optimal proportional growth program? (2) Is there any tendency for optimal economic programs to approximate proportional growth programs in the long run? Our current knowledge of the answers to these questions is far from complete; the results reviewed in Chapter V would suggest the following tentative conclusions: (1) For an optimal proportional growth program, the shadow rate of interest will be at least as large as the rate of growth. (2) An optimal program will typically tend towards proportional growth in the long run, provided there are no primary resources in the economy, or provided all primary resources grow at the same (constant) rate. It should be emphasized that so far theorems of this type have been proved only under fairly special assumptions, including the assumptions of constant returns to scale, and constant technology.

I. DESCRIPTION OF AN ECONOMIC SYSTEM IN TIME

1. Introduction

Any precise discussion of economic planning must take place in a context in which the alternative paths of economic development are precisely described. Therefore this first chapter is devoted to a review of some of the more important theoretical models of an economy in time. The emphasis will be on describing the production and consumption possibilities, especially the former. Consumption will be discussed again in the next chapter, which is devoted to the problem of describing preferences among alternative economic programs.

A model of an economy in time should be capable of describing, in addition to the usual features of a static model, such phenomena as durability, aging, storage, waiting, as well as the sequential aspects of production, from raw materials and labor, through investment goods and intermediate goods, to consumption goods. A special case of importance is that of education.

In my opinion, the most general and potentially useful framework is the one in which one tries to describe the possibilities of transforming the economy from one period to the next. This approach has a minimum of conceptual problems, and lends itself most easily to technical measurement. Other models, based upon the ideas of "waiting" or "gestation" have, for me, an element of mystery, unless based in turn upon a model of technological transformation.

Finally, one wants to be able to describe technological change and learning. In other words, having described the technological possibilities for the evolution of an economy, one may go a step further and try to describe the ways in which these technological "laws" change in time.

One technical point to be mentioned is that I have chosen the discrete time - or period analysis - approach in these notes, as opposed to the use of continuous time. The former is conceptually simpler, and can also be used with more elementary mathematical tools.

2. Commodities

We start with a fixed list of commodities, numbered 1 to M. The concept of "commodity" is to be interpreted rather generally, including capital goods, intermediate goods, consumer goods (in so far as these distinctions make sense), land of different types, and labor of different types and skills. Goods are to be distinguished by their physical qualities, including age and location. Indeed, all distinctions that could be important from the point of view of production, consumption and trade are, in principle, to be embodied in the classification used.

It should be pointed out that the assumption of a finite list of commodities implies that if the other physical qualities of a commodity change with age, then that commodity cannot last forever. It should also be mentioned that certain models of technological change require, in essence, an infinite list of commodities.

3. Production and Consumption Possibilities

Suppose that at the beginning of any given period, there is available a stock z_i of commodity i. A certain quantity, c_i, is devoted to consumption, and the rest, x_i, is used as an input into the productive process (including use as inventory, stock of machines, etc.). Given z_i, one must of course have

$$c_i + x_i = z_i \,, \qquad c_i \geqq 0,\ x_i \geqq 0.$$

If c denotes the vector with components c_i, etc., then one can rewrite the above as

$$(3.1) \qquad c + x = z \,, \qquad c \geqq 0,\ x \geqq 0.$$

Given the input vector x, i.e. having determined the gross allocation between consumption and production, it remains to determine how to use x in the productive process. The outcome or result of the productive process, at the end of the period in question, will be some vector y of quantities of commodities, the output vector.

To the output vector y may possibly be added a vector q of quantities of commodities made available exogenously (primary resources, grants from outside the economy, etc.). The resulting sum (y+q) is then available as the new initial stock vector for the succeeding period.

Given any input x, only certain outputs y are technologically possible. One says that an input-output pair (x,y) is feasible if it is possible to produce y from x in one period. We may denote the set of all feasible input-output pairs by the symbol $\mathcal{T}$. The set $\mathcal{T}$ is sometimes called the technological transformation set or production possibility set.

The concept "production" is to be interpreted very generally, including in principle all transitions of the state of the system that are economically interesting. In particular, it includes the phenomena of storage and aging, with or without associated physical changes.

Indeed, for certain purposes it may be useful to treat "consumption" itself as part of the productive process, just as we treat the consumption of corn by hogs. However, in most present-day societies a purely technological treatment of consumption would not seem appropriate.

At the other extreme, one often divides goods into "consumer goods" and "investment and production goods", so that in such an approach many components of the vectors c and x would be zero (and not the same ones!). However, in principle, any commodity can be used for both consumption and production (e.g. automobiles).

4. Economic Programs

We consider now a sequence of periods t = 1, 2, ..., T (where T may be finite or infinite), with a given initial stock vector z(1). Also, in every period $t \geqq 2$, a vector $q(t) \geqq 0$ is fed into the system exogenously.

A feasible program is a sequence of T quadruples

$$[z(t),\ c(t),\ x(t),\ y(t)]$$

satisfying the technological and accounting constraints:

$$c(t) + x(t) = z(t),$$
$$c(t) \text{ and } x(t) \geqq 0, \qquad t = 1, \ldots, T,$$
$$[x(t), y(t)] \text{ in } \mathcal{T},$$

and

$$z(t) = y(t-1) + q(t), \qquad t = 2, \ldots, T.$$

An economic program may be wholly or partly planned, or wholly determined by autonomous behavior. For example, consumption $c(t)$ may be determined as a function of $z(t)$ by a free market system (as described by demand functions, Engel curves, consumption functions, etc.), whereas production $y(t)$ may be centrally planned, as a function of $x(t) = z(t) - c(t)$.

In the following sections I review some special models of production possibilities.

5. Linear Activity Analysis Model of Production Possibilities+

Suppose that there are N production activities $j = 1, \ldots, N$, and denote the "level" of activity j by $a_j \geqq 0$. Both input and output vectors are linear transformations of the activity vector a, thus:

$$(5.1) \qquad \begin{aligned} x_i &= \sum_{j=1}^{N} r_{ij} a_j \\ y_i &= \sum_{j=1}^{N} p_{ij} a_j \end{aligned} \qquad i = 1, \ldots, M.$$

In matrix notation we have

$$(5.2) \qquad \begin{aligned} x &= Ra, \\ y &= Pa. \end{aligned}$$

+See KOOPMANS, 1951, 1957. A list of references is placed at the end of each chapter, and a combined list is placed at the end of the book.

It is usual to take account of the possibilities of "throwing away" goods, or of unemployment, by replacing the equality signs in (5.2) by inequality signs:

$$(5.3)\qquad \begin{aligned} x &\geqq Ra, \\ y &\leqq Pa. \end{aligned}$$

The coefficients r_{ij} and p_{ij} are assumed to be $\geqq 0$. The coefficient r_{ij} is the input of commodity i required to operate activity j at unit level. The coefficient p_{ij} is the output of commodity i when activity j is operated at unit level.

The set $\mathcal{T}$ of feasible input-output pairs is the set of pairs (x,y) that satisfy (5.3) for some non-negative vector a of activity levels.

Condition (5.3) implies, but is not implied by, the following condition:

$$(5.4)\qquad y \leqq x + (P-R)a.$$

For example, if all of the elements of (P-R) were non-negative, and some were positive, then (5.4) could be satisfied by a positive output vector y, with a zero input vector x. The economic interpretation of the sense in which (5.3) is stronger than (5.4) is that the inputs $x \geqq Ra$ must be available at the beginning of the period in which the output $y \leqq Pa$ is produced.

The Linear Activity Analysis (LAA) model is a very general one, except for the finiteness of the number of activities. It is well adapted to statistical measurement, and in the computation of economic programs one is led to the techniques of linear programming.

The following example is meant for expository purposes only.

Example: "Shoe Production"

Suppose there are 5 commodities: labor, leather, shoes, new machines, and second-hand machines. A machine lasts for only two periods. Let activity 1 be the production of shoes using new machines.

Commodity	Input Coefficients r_{i1}	Output Coefficients p_{i1}
1. Labor	$r_{11} = 2$	$p_{11} = 0$
2. Leather	$r_{21} = 1000$	$p_{21} = 0$
3. Shoes	$r_{31} = 0$	$p_{31} = 500$
4. New Machines	$r_{41} = 1$	$p_{41} = 0$
5. Second-hand Machines	$r_{51} = 0$	$p_{51} = 1$

Let activity 2 be production of shoes using second-hand machines.

Commodity	Input Coefficients r_{i2}	Output Coefficients p_{i2}
1. Labor	3	0
2. Leather	1000	0
3. Shoes	0	400
4. New Machines	0	0
5. Second-hand Machines	1	0

If two new machines were provided every year, one could have two new and two used machines in every period, so that in every period one could use the activity vector $\binom{2}{2}$. Then inputs would be

$$Ra = \begin{pmatrix} 2 & 3 \\ 1000 & 1000 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \binom{2}{2} = \begin{pmatrix} 10 \\ 4000 \\ 0 \\ 2 \\ 2 \end{pmatrix} ,$$

and outputs would be

$$Pa = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 500 & 400 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \binom{2}{2} = \begin{pmatrix} 0 \\ 0 \\ 1800 \\ 0 \\ 2 \end{pmatrix} .$$

Unless labor, leather, and new machines were provided exogenously, column vectors describing their production would have to be added to the input and output matrices R and P.

Note that for convenience I have chosen to measure each activity by the number of machines used. This was arbitrary. I might just as well have chosen any other unit, e.g., number of shoes produced.

The LAA model easily expresses joint production.

Example: "Meat Packing"

Commodity	Input Coefficients	Output Coefficients
1. Labor	1	0
2. Cattle	5	0
3. Meat	0	2500
4. Hides	0	10
5. Bone	0	50

In this example there is only one activity, so that R and P are each (5 x 1).

Storage of a commodity can be expressed by an activity with identical input and output coefficients:

$$\begin{array}{cc} \text{input} & \text{output} \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} ;$$

or, if there were 5 percent loss in storage, one would have

$$\begin{array}{cc} \text{input} & \text{output} \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0.95 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} .$$

The LAA model has the property of constant returns to scale; i.e., if (x,y) is a feasible input-output pair, then so is $(\lambda x, \lambda y)$ for any non-negative number λ. This is, of course, achieved by multiplying all activity levels by λ.

In many cases, one may only be able to multiply by scale factors λ that are whole numbers. This would presumably be the case in the example "Shoe Production", in which the number of machines used must typically be an integer, so that the activity levels must be integers. However, even in such cases it may be convenient to make the approximation of assuming that any non-negative number is a possible activity level.

The LAA model also has the property of convexity or non-increasing marginal productivity; i.e., if (x,y) and $(\bar{x},\bar{y})$ are feasible input-output pairs, then so is

$$\underline{(\alpha x + [1-\alpha]\bar{x},\ \alpha y + [1-\alpha]\bar{y})}\ ,$$

for any α with $0 \leqq \alpha \leqq 1$. This is achieved, of course, by using the activity level vector $\alpha a + (1-\alpha)\bar{a}$, where a and $\bar{a}$ are the activity level vectors corresponding to (x,y) and $(\bar{x},\bar{y})$, respectively.

Exercise. Construct a hypothetical LAA model with the following properties:

Commodities: land, labor, machine tools, agricultural machinery, food. Assume that machine tools last 3 periods, and that agricultural machinery lasts 2 periods. Assume that

a) machine tools are produced from labor,
b) agricultural machinery is produced from labor and machine tools,
c) food is produced from land and labor, or from land, labor, and agricultural machinery.

Provide two alternative activity vectors for the production of food from land, labor, and agricultural machinery.

If land and labor are given exogenously in each period, this induces certain constraints on the possible activity levels; what form do these constraints take?

6. Dynamic Input-Output Model (Leontieff)

The so-called Dynamic Input-Output (DIO) model is a special case of the Linear Activity Analysis model presented in the last section. In the DIO model there is a one-to-one correspondence between commodities and activities, and each activity can be regarded as the activity of producing the corresponding commodity. Thus the DIO model describes a situation in which there is no joint production. Also, one says that there are fixed input proportions for the production of any one commodity.

Two kinds of input requirements are distinguished in the DIO model: "flow" requirements and "stock" requirements. Although stocks enter the model, durable goods are not distinguished according to age, as is possible in the more general LAA model.

One can consider two alternative versions of the DIO model.

Version I.

To produce 1 unit of commodity j requires a_{ij} units of commodity i, which amount is used up in the production process during the current period. One requires in addition b_{ij} units of commodity i, which amount is not used up, but is conserved as a stock. Thus:

$$(6.1)\qquad \begin{aligned} R &= A + B, \\ P &= I + B, \end{aligned}$$

where $A = ((a_{ij}))$, $B = ((b_{ij}))$, and I denotes the identity matrix. The elements of B are called the "capital-output coefficients".

Version II.

In the second version, the flow requirements are entirely met from current production, i.e. within the period in question, so that

$$(6.2)\qquad \begin{aligned} R &= B, \\ P &= I - A + B. \end{aligned}$$

In both versions, if stocks suffer physical depreciation in the form of loss (but not in the form of changed physical characteristics), then this phenomenon can be described by replacing the terms b_{ij} in the output matrices by the terms $d_{ij}\ b_{ij}$, where

d_{ij} represents the depreciation factor, per period, for commodity i when used in the production of commodity j. In matrix notation, let $\bar{B} = ((d_{ij}b_{ij}))$; then one has

Version I
$$R = A + B$$
$$P = I + \bar{B}$$

Version II
$$R = B$$
$$P = I - A + \bar{B}$$

7. Linear-Logarithmic Production Functions (Cobb-Douglas)

In the model to be described in this section, one retains the assumption of no joint production for new goods, but, on the other hand, one allows for the possibility of variable input proportions, and is able to describe the aging of durable goods. The model is a multisector generalization of the model used by Cobb and Douglas to describe the productivity of labor and capital. In particular, the logarithm of output of each new commodity is a linear function of the logarithms of the inputs into that industry.

Let the commodities be divided into two groups, new (i.e. newly produced) ($i = 1,\ldots,N$), and second-hand ($i = N+1,\ldots,M$).

Production of New Commodities. Let x_{ij} be the quantity of commodity i devoted to the production of commodity j, and let y_j be the output of commodity j; then

$$\log y_j = \beta_j + \sum_{i=1}^{M} \alpha_{ij} \log x_{ij}, \quad j = 1, \ldots, N. \tag{7.1}$$

Assume $\alpha_{ij} \geq 0$. Further, the assumption of constant returns to scale, if appropriate, can be expressed by

$$\sum_{i} \alpha_{ij} = 1, \qquad j = 1, \ldots, N. \tag{7.2}$$

If x_i is the total amount of commodity i devoted to (all) production, then

$$(7.3) \qquad \sum_{j=1}^{N} x_{ij} = x_i .$$

The production function (7.1) can be rewritten

$$(7.4) \qquad y_j = (e^{\beta_j}) \prod_{i=1}^{M} x_{ij}^{\alpha_{ij}} ;$$

or, to take account of the possibility of disposal, with an inequality $\leq$. Note that both new and second-hand inputs typically enter as inputs into each function (7.4).

Aging and Depreciation. Assume that each commodity is used up at a rate that depends upon the commodity and upon its age, but not upon the use to which it is put. (In this model, commodities can be distinguished by age.) If j represents a second-hand commodity, then there is some other commodity in the list, say j', that represents the same good of age one period less. Assume that

$$y_j = (e^{\beta_j}) x_{j'} ,$$

or

$$(7.5) \qquad \log y_j = \beta_j + \log x_{j'} , \qquad j = N + 1, \ldots, M .$$

Thus one can express an arbitrary age pattern of physical loss of "durable" goods, in other words, an arbitrary distribution of length of life.

Example: If commodity 4 is a new machine, and commodity 5 is a 1-period-old machine of the same type, then one might have

$$y_5 = (0.6) x_4 ,$$

expressing the fact that 60 percent of the new machines survive to the second period (here $\beta_5 = \log 0.6$). Furthermore, if no machines of age more than one period appear on the list, this expresses the fact that no machines of this type last more than two periods.

A Unified Notation. One can simplify the notation, and express the production of new and old goods in a unified way as follows.

Let

$$x_{ij} = f_{ij}x_i, \quad i = 1, \ldots, M, \; j = 1, \ldots, N, \tag{7.6a}$$

and for $j = N + 1, \ldots, M$ define

$$f_{ij} = \alpha_{ij} = \begin{cases} 1 & \text{if } i \text{ is the predecessor of } j \text{ from the point of view of age (i.e. } i=j') \\ 0 & \text{otherwise.} \end{cases} \tag{7.6b}$$

Also define[+]

$$\left\{ \begin{array}{l} Y_j = \log y_j, \quad X_i = \log x_i \\ \left.\begin{array}{l} A = ((\alpha_{ij})) \\ f = ((f_{ij})) \end{array}\right\} \quad i,j = 1, \ldots, M \\ \eta_j(f) = \sum_i \alpha_{ij} \log (f_{ij}), \\ \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_M \end{pmatrix}, \; X = \begin{pmatrix} X_1 \\ \vdots \\ X_M \end{pmatrix}, \; Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_M \end{pmatrix}, \; \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix}, \end{array} \right. \tag{7.6c}$$

where single brackets denote vectors and double brackets denote matrices. Then the production of both new and old commodities is expressed by

[+]To avoid indeterminacy, make the convention that $0 \cdot \log 0 = 0$.

(7.7) $$Y = \beta + \eta(f) + A'X \ .$$

Note that f_{ij} is the proportion of the input of commodity i that is devoted to production of commodity j (not to be confused with input proportions for a given industry!). Hence

(7.8) $$f_{ij} \geq 0 \ , \quad \sum_{j=1}^{N} f_{ij} = 1 \ , \qquad i = 1, \ldots, M \ .$$

Note, too, that $\eta_j(f) = 0$ for $j = N + 1, \ldots, M$, i.e. for all of the second-hand commodities.

8. Constant Elasticity of Substitution Production Function (Arrow-Chenery-Minhas-Solow)

In the production of any single commodity, the Dynamic Input-Output model provides for no substitution of one input factor for another. On the other hand, the Linear-Logarithmic production function provides for substitution, but of a special form. Formally, the elasticity of substitution between any two inputs (see ALLEN, pp. 340-343) is zero in the DIO model, and one in the LL production function. A general class of production functions has recently been proposed with the property of an arbitrary constant elasticity of substitution, and including the fixed-proportions and linear-logarithmic production functions as special cases.

This more general function promises to be of interest, but it has not yet been incorporated in any planning model, and so will not be discussed here.

9. Technical Change

One may think of technical change as a change in the production possibility set $\mathcal{T}$, or, more generally, as a change in both the list of commodities and $\mathcal{T}$. In terms of the special models discussed above, the first type of change would be expressed by changes in the input and output coefficients, or by changes in the parameters of the production functions. The second, more extensive, type of change would involve the introduction of new goods, new equipment, and new activities or production functions.

We typically think of technical change in terms of <u>improvement</u>; e.g., this may take the form of a decrease in some input coefficient or an increase in some output coefficient. Of course, we also hear of examples of technical "decline", e.g., the so-called "lost arts" of making fine swords or stained glass windows.

<u>Neutral Technical Change</u>. One calls technical change of the first type <u>neutral</u> if, roughly speaking, it affects "equally" the productivities of the various input factors. For example, if an activity produces one output, and that output coefficient is increased, whereas the input coefficients are left unchanged, then the resulting technical change is neutral.

If one is using a production function model, let the production function for a particular commodity be

$$y = bg(x_1, \ldots, x_M); \tag{9.1}$$

then a change in b might be called neutral.

A more specific meaning of neutrality is the following. Let $\hat{x}_1, \ldots, \hat{x}_M$ be the input quantities that minimize the cost of producing a given quantity of output $\hat{y}$, at given input prices $p_1, \ldots, p_M$. Now consider a change in the production function, and look for the new minimum cost inputs to produce the same quantity $\hat{y}$, with prices unchanged. The technical change is called neutral if the new cost-minimizing input quantities are proportional to the old ones.

If the production function in (9.1) exhibits constant returns to scale, i.e., if the function f is homogeneous of degree 1, then a change in the parameter b will be neutral in the more specific sense just defined.

<u>Improvement of Capital Equipment</u>. Even without any change in the list of consumption goods, the most widely held view currently is that advances in technical knowledge about the production of consumer goods are largely embodied in new capital equip-

ment — and in the corresponding newly required labor skills. This involves additions to the list of commodities, and corresponding additions to the list of activities.

As an example, I sketch a model used by ARROW. Consider the following "commodities":

aggregate output

labor

capital goods of type 1, 2, 3, etc., ad inf.

Imagine that capital goods of type t are built at time t, and embody the "latest improvements" in technology. One unit of capital of type t

a) requires $L(t)$ units of labor input,

b) yields $P(t)$ units of output,

c) has a given lifetime.

Technical improvement might be expressed by:

$L(t)$ decreasing or constant,

$P(t)$ increasing.

Arrow makes the special assumptions:

$L(t)$ constant

$P(t) = pt^{-n}$,

where p and n are positive constants. The assumption about $P(t)$ is suggested by some experience in the U.S. aircraft industry.

REFERENCES TO CHAPTER I

K. J. ARROW, "The economic implications of learning by doing," Institute for Mathematical Studies in the Social Sciences, Stanford University, ONR Technical Report No. 101, Dec. 7, 1961.

K. J. ARROW, H. B. CHENERY, B. S. MINHAS, and R. M. SOLOW, "Capital-labor substitution and economic efficiency," Review of Economics and Statistics, 43 (1961), 225-250.

P. A. DOUGLAS, "Are there laws of production?" American Economic Review, 38 (1948), 1-41.

L. JOHANSEN, A Multisectoral Model of Economic Growth, Amsterdam: North - Holland, 1960.

L. JOHANSEN, "Substitution vs. fixed production coefficients in the theory of economic growth: a synthesis," Econometrica, 27 (1959), 157-176.

T. C. KOOPMANS (ed.), Activity Analysis of Production and Allocation, New York: Wiley, 1951.

T. C. KOOPMANS, Three Essays on the State of Economic Science, Chapter I, New York: McGraw-Hill, 1957.

W. W. LEONTIEFF, Studies in the Structure of the American Economy, Oxford: Oxford University Press, 1953.

E. MALINVAUD, "Capital accumulation and efficient allocation of resources," Econometrica, 21 (1953), 233-268.

R. M. SOLOW, "Investment and technical progress" in Arrow, Karlin, and Suppes (eds.), Mathematical Methods in the Social Sciences, Stanford: Stanford University Press, 1960, pp. 89-104.

R. M. SOLOW, "Technical change and the aggregate production function," Review of Economics and Statistics, 39 (1957), 312-320.

J. VON NEUMANN, "A model of general economic equilibrium," Review of Economic Studies, 13 (1945).

CHAPTER II

BASIC CRITERIA FOR CHOOSING AMONG ECONOMIC PROGRAMS

1. What is Important about an Economic Program?

In developing criteria for choice among alternative economic programs, one may ask two questions:

1) What aspects of the program are to be looked at in making a choice?
2) Having decided which are the important aspects, how precisely are they to be combined and evaluated?

Current discussion of economic planning seems to concentrate on the following basic aspects (basic from the point of view of preference among programs):

i) Total or per-capita income
ii) Composition of income
iii) Distribution of income
iv) Employment

One is, of course, concerned with the time pattern of all of these.

One might be tempted to include a fifth aspect when one is discussing programs with a finite horizon, namely, the terminal stocks that are to be carried forward at the end of the program. This aspect is not basic, however, in the same sense as the above four are, since the value attached to such terminal stocks is typically derived from their power to produce income in the period beyond the horizon.

Under aspects (i) and (ii), one looks at the sequence of vectors of quantities of commodities designated for consumption in each period (denoted by $c(t)$ in Chapter I), with or without dividing by the population number. If population is determined exogenously, i.e. is not affected by the choice of an economic plan, then increasing total income and increasing per-capita income are equivalent. But if population growth (or decline) depends upon the program chosen, then population must typically enter the list of "commodities", and maximizing total and per-capita income will typically not be equivalent. Indeed, maximizing per-capita income may lead to (possibly) socially unacceptable programs, involving, for example, restricting births, or even killing a part of the population!

The remaining sections of this chapter are devoted to an exposition of some means of describing or expressing preferences among alternative sequences of consumption. However, a word should be said here about a certain basic difficulty in arriving at a satisfactory definition of consumption. The models described in Chapter I have in common the feature that the activities of consumption and production are in a certain sense independent — from a mathematical point of view one might say they are additive. More precisely, recall that the beginning-of-period stock z_i of each commodity is divided into two parts, c_i and x_i, the quantity c_i being "consumed", and the quantity x_i being used as an input into production. In particular, one has the accounting identity,

$$c_i + x_i = z_i \ .$$

It seems to me doubtful, however, that in every case one can achieve such an algebraic separation of consumption and production. For example, the consumption of food is valued in itself by many consumers, but at the same time variations in the consumption of food may well affect the productivity of labor, and therefore be properly included in the vector x of inputs. In spite of this type of difficulty, the assumption of the separability of consumption and productive inputs will be retained in these notes, in view of its consistency with conventional income accounting procedures, and in the hope that the resulting errors are not too significant.

Employment goals are typically connected with income distribution goals. On the one hand, it is usual to find workers struggling to obtain shorter working hours (for the same pay, of course). On the other hand, it is very likely, in Western cultures at least, that most people would want <u>some</u> employment, even if such employment were not necessary to obtain an acceptable income! If leisure is regarded as non-productive consumption of a stock of labor, then this last point is a special case of the preceding paragraph.

However, in the present, far-from-Utopian state of the world, unemployment is considered bad primarily because of the resulting low incomes of the unemployed. Unemployment is also considered bad because it is thought of as wasteful. But this evaluation is "derived" rather than "basic" in the sense that there may be situations in which maximization of total income requires some unemployment. In other words, whether or not unemployment is wasteful depends upon the particular situation.

In the remainder of these notes (with the exception of Section 2 of Chapter V) I will confine my attention to criteria that compare alternative economic programs solely on the basis of comparisons of the corresponding sequences c(t) of vectors of total consumption. This means, in particular, that I will be ignoring explicit consideration of distribution or employment goals.

2. Efficiency

The criterion of efficiency is about the weakest (i.e. least selective) useful criterion that is generally proposed for the evaluation of economic programs. A program is efficient if consumption of any commodity in any period cannot be increased without decreasing the consumption of some other commodity in that period, or decreasing the consumption of some commodity in some other period.

Formally, if c and d are two vectors, write

$$
\begin{aligned}
& c \geqq d, \text{ if } c_i \geqq d_i \text{ for all } i; \\
(2.1) \qquad & c > d, \text{ if } c \geqq d \text{ but } c \neq d; \\
& c \gg d, \text{ if } c_i > d_i \text{ for all } i.
\end{aligned}
$$

A feasible program with a sequence of consumption vectors c(1),...,c(T), is efficient is there is no other feasible program with consumption vectors, say, c(1),...,c'(T) such that

$$
(2.2) \qquad \begin{aligned}
& c'(t) \geqq c(t) && \text{for all } t, \text{ and} \\
& c'(t) > c(t) && \text{for some } t.
\end{aligned}
$$

Typically, the class of efficient programs will be very large, and some further criteria will be needed to arrive at a choice among the efficient programs.

Since efficiency is a generally agreed-upon criterion, it is not surprising that economic theorists have devoted a good deal of attention to the problem of characterizing efficient programs. Some of these results will be reviewed in Chapters III and IV.

3. Social Welfare Functions and Social Time Preference

In this section I describe some special formulations of preference among alternative time patterns of consumption, in terms of numerical functions of the sequence of consumption vectors $c(1)$, ..., $c(T)$. I will call a social welfare function any numerical function U defined on the set of possible sequences $c(1), \ldots, c(T)$ such that

$$U[c(1), \ldots, c(T)] > U[c'(1), \ldots, c'(T)]$$

expresses the fact that the sequence $c(1), \ldots, c(T)$ is preferred (e.g. by the planner) to the sequence $c'(1), \ldots, c'(T)$.

One Period Welfare. The first special assumption that suggests itself is that the social welfare that is derived from a sequence of consumption vectors can be expressed as a function of one-period welfares. In other words, suppose that in each period t one can define the welfare (or "income") that is attributable to the consumption $c(t)$ in that period only, say

$$v_t = u_t[c(t)] \ , \tag{3.1}$$

and suppose that the social welfare function for the sequence $c(1), \ldots, c(T)$ is defined in terms of the numbers v_t, thus:

$$U[c(1), \ldots, c(T)] = V(v_1, \ldots, v_T) \ . \tag{3.2}$$

In this case, the function u_t expresses how the quantities of different commodities consumed in period t are combined to provide a single measure of welfare in that period, whereas the function V describes the preferences among alternative time-patterns of one-period welfare.[+]

I first describe some special forms of one-period social welfare functions.

(i) Linear Case. Suppose u has the form

$$u(c) = u(c_1, \ldots, c_M) = \sum_{i=1}^{M} \omega_i c_i \ . \tag{3.3}$$

This is the form taken by most index numbers. Thus the "weights" ω_i may be constant prices, to give an index of "real income".

Note that a proportional change in all the components of c results in an increase of welfare in the same proportion; i.e.,

$$u(kc) = ku(c) \ , \qquad \text{for any number } k.$$

(ii) Linear-Logarithmic Case. Suppose u has the form

$$u(c) = \sum_{i=1}^{M} \omega_i \log c_i \ . \tag{3.4}$$

This is, of course, defined only if c_i is positive for all i for which ω_i is non-zero.[++]

A proportional change in all the components of c adds a constant amount of welfare, thus

---

[+] Note that a strictly increasing monotonic transformation of the functions U and V does not change the order of preference among sequences of consumption or one-period welfare. The same is typically not true of monotonic transformations of the one-period welfare function u, since the numerical values taken by the function u enter into the function V.

[++] Here, and elsewhere in these notes, I use the convention that $0 \cdot \log 0 = 0$.

$$u(kc) = \sum_i \omega_i \log (kc_i)$$

$$= \sum_i \omega_i(\log k + \log c_i)$$

$$= (\sum_i \omega_i)\log k + \sum_i \omega_i \log c_i$$

$$= (\sum_i \omega_i)\log k + u(c) .$$

The linear-logarithmic welfare function exhibits <u>decreasing marginal welfare</u>, since

$$\frac{\partial u(c)}{\partial c_i} = \frac{\omega_i}{c_i} .$$

The linear-logarithmic function is the logarithm of a <u>weighted geometric mean</u> of the consumptions of the individual commodities, in the case $\omega_i \geq 0$ and $\sum_{i=1}^{M} \omega_i = 1$, since

$$e^{u(c)} = \prod_{i=1}^{M} c_i^{\omega_i} .$$

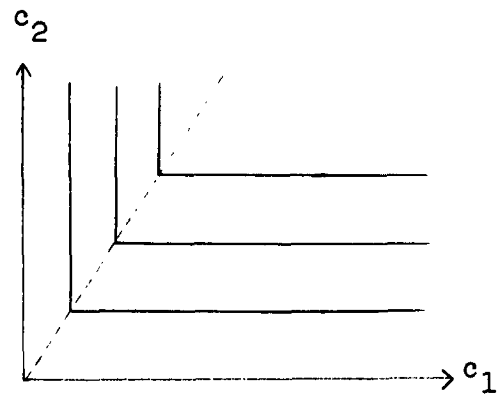
(iii) <u>Desired Proportions</u>. Suppose that one wishes to maximize consumption, but in certain <u>desired proportions</u>, so that consumption of any individual commodity in excess of the desired proportions is not valued. Let $\omega_1, \ldots, \omega_M$ be the desired proportions of the M commodities. If $\omega_i > 0$ for every i, then this one-period welfare function can be expressed as

$$(3.5) \qquad u(c) = \min_i \left(\frac{c_i}{\omega_i}\right) .$$

More generally, denoting by ω the vector with coordinates ω_i, if $\omega \geq 0$, then

$$(3.6) \qquad u(c) = \min \{ k | c \geq k\omega \} .$$

The figure shows typical "iso-welfare" contours for the case of two commodities.



The dotted line in the figure indicates the set of consumption pairs (c_1, c_2) that are in the desired proportions.

This type of criterion function is often used in the Soviet Union (see WARD).

Intertemporal Preference. Once having chosen a measure of one-period welfare, it remains to choose a way of expressing preference among alternative sequences v_t of one-period welfare values. In other words, it remains to choose a particular form for the function V of equation (3.2), which might be called the social intertemporal preference function.

The simplest function that suggests itself is a sum of one-period welfares:

$$V(v_1, v_2, \ldots) = \sum_t v_t \ .$$

More generally, one might consider a linear function of one-period welfares:

$$V(v_1, v_2, \ldots) = \sum_t d_t v_t \ . \tag{3.7}$$

A special case of (3.7) that is of some appeal is produced by taking

$$d_t = d^t \ , \tag{3.8}$$

where d is a given positive number; in this case (3.7) becomes

$$(3.9) \qquad V(v_1, v_2, \dots) = \sum_t d^t v_t \ .$$

The number d is called the <u>social time discount factor</u>, or <u>time preference factor</u>. If $d < 1$, this expresses a preference for present as against future consumption; inversely, $d > 1$ expresses a preference for future as against present consumption.

If the number of periods is infinite, the sums (3.7) and (3.8) may not converge. In particular, if the sequence v_t grows at least as fast as the sequence $(1/d_t)$, then the sum (3.7) will be infinite.

Other criteria focus on the long-run, or <u>asymptotic</u>, behavior of the sequence v_t. For example, one can take

$$(3.10) \qquad V(v_1, v_2, \dots) = \lim_{t\to\infty} v_t \ ,$$

if this limit exists. This is essentially the criterion applied when one looks for best <u>stationary states</u>.

In the typical economic growth problem, the sequence v_t grows without limit (even if v_t represents per-capita income), and the formulation (3.10) is not useful. In such a case one may consider the <u>rate of growth</u>. The rate of growth r_t of the sequence v_t at period t is defined as

$$(3.11) \qquad r_t = \left(\frac{v_t}{v_{t-1}} \right) - 1 \ .$$

(I am assuming that all of the v_t are positive.) If

$$(3.12) \qquad r = \lim_{t\to\infty} r_t$$

exists, it is called the <u>asymptotic</u> or long-run <u>rate of growth</u>.[+]

[+]<u>Note</u>. If time is continuous, and v_t is a differentiable function of time, then the instantaneous growth rate of v_t at time t is defined by $\frac{dv_t}{dt}/v_t$. Thus the growth rate of $v_t = ae^{bt}$ is b.

In <u>exponential growth</u>

$$v_t = ab^t ,$$

the growth rate is constant and equal to (b-1).

The asymptotic rate of growth of a sequence may be zero, even though the sequence grows without limit, e.g. for

$$v_t = t .$$

Two sequences may have the same asymptotic growth rate (or same limit) and yet one be always larger than the other. More troublesome is the case in which two sequences have the same asymptotic growth rate, but the first is greater than the second at some time periods, and smaller in others.

The last difficulty suggests a different type of criterion. Instead of trying to assign a numerical value expressing "social welfare" to every sequence, one may be satisfied with <u>pair-wise comparisons</u> of sequences, for example by assigning a numerical value to the sequence of differences.

Finally, one may combine several criteria hierarchically, or one may try to maximize the value of one criterion, subject to constraints on the value of another.

REFERENCES TO CHAPTER II

S. CHAKRAVARTY, "The existence of an optimum savings program," <u>Econometrica</u>, <u>30</u> (1962), 178-187.

A. G. PAPANDREOU, "Planning Resource Allocation for Economic Development," Center of Economic Research, 1962.

B. WARD, "Kantorovich on economic calculation," <u>Journal of Political Economy</u>, <u>68</u> (1960), 545-556.

CHAPTER III. DERIVED CRITERIA

1. Present Value

1.1 Introduction

Economists have long discussed the possible role of prices in promoting efficient production and allocation in a market economy. More recently, there has been considerable study of the possibilities of using "imaginary prices" as aids in the calculation of good economic programs in cases in which the economic decisions are not left to a market. In this section I will review some of the theorems on the connection between economic optima and such systems of imaginary prices — usually called shadow prices in the technical literature.

Essentially, these theorems indicate how suitably chosen shadow prices can be used to test an economic program for optimality in the sense of some of the basic criteria discussed in the last chapter. Given a shadow price $p_i(t)$ for each commodity in each period, one can calculate the total "shadow value" of any sequence of consumption vectors $c(t)$ as

$$\sum_{i,t} p_i(t)\ c_i(t) .$$

One can also calculate the "shadow profit" for any input-output pair $[x(t),y(t)]$ as

$$\sum_i p_i(t+1)\ y_i(t) - \sum_i p_i(t)\ x_i(t) ,$$

and add up these shadow profits for any program of production. If one interprets the shadow prices corresponding to future periods as discounted prices (and it will be seen that this is appropriate), then one can interpret a total value of the type just described as a present value. The three theorems I will discuss are, roughly speaking:

(1) An economic program is optimal if and only if the corresponding sequence of consumption has maximum present value among all feasible consumption sequences, provided the present value is calculated using suitably chosen shadow prices.

(2) An economic program is optimal if and only if, for suitably chosen shadow prices,

(a) no other consumption sequence (feasible or not) with the same, or less, present value is preferable, and

(b) no other feasible plan of production yields a higher present value of total profits.

(3) If production can be divided into several sectors, with no external economies or diseconomies between sectors, then result (2) above can be extended to apply to separate profit calculations by the individual sectors.

Naturally these theorems hold only under certain assumptions about the production possibilities and about the criterion of optimality.

The theorems are interesting for at least two reasons. First, they may be used to form the basis of a method of calculating optimal programs. Second, theorem (3) suggests a way to decentralize the process of economic calculation or decision making.

1.2 An Example

For those who are not familiar with the concept of prices as indicators of optimality, it may be useful to begin with a simple example of calculating an optimal one-period production plan.

Suppose that two goods — labelled 1 and 2 — are to be produced, and that two other goods — call them "capital" and "labor" — are used as inputs. Suppose further that the production function for good 1 is

$$y_1 = (e^{\beta_1}) K_1^{\alpha_1} L_1^{(1-\alpha_1)} , \tag{1.1}$$

where

y_i is the output of good i,

K_i is the quantity of capital devoted to the production of good i,

L_i is the quantity of labor devoted to the production of good i,

and β_i and α_i are parameters, with $\alpha_i > 0$. Let K and L be the total amounts of capital and labor, respectively, that are available as inputs into production, so that

$$(1.2) \qquad K_1 + K_2 \leqq K , \qquad L_1 + L_2 \leqq L ,$$
$$K_i \geqq 0 , \qquad L_i \geqq 0 .$$

The set of feasible output pairs (y_1, y_2) is represented in Figure 1 by the set of points bounded between the curve and the two co-ordinate areas. The <u>efficient</u> output pairs lie on the curve itself, which we may call the efficiency curve.

To illustrate result (1) of Section 1.1, consider the efficient+ output pair $(\hat{y}_1, \hat{y}_2)$ in Figure 2.

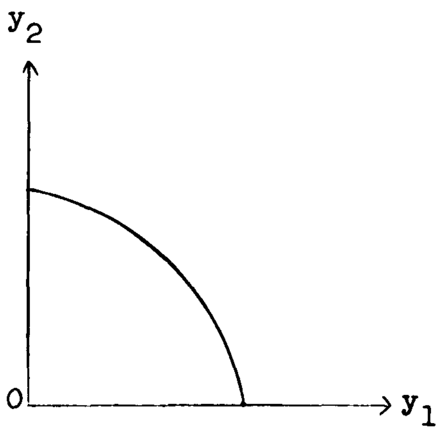


Figure 1

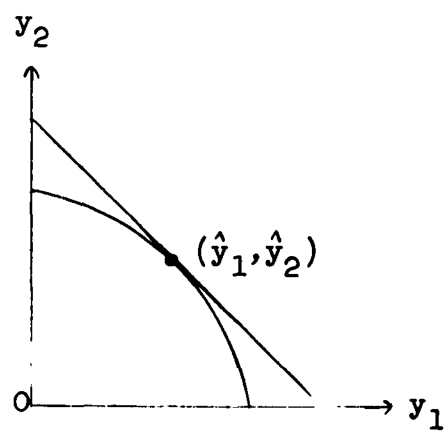


Figure 2

+In Chapter II the criterion of efficiency was applied to consumption. It can be extended to production here in an obvious way, or alternatively the reader may imagine here that all of the output will be consumed.

Through the point (y_1,y_2) draw a line tangent to the efficiency curve (see Figure 2), and denote the equation of that line by

$$(1.3) \qquad p_1y_1 + p_2y_2 = d.$$

The parameter d is, of course, given by

$$(1.4) \qquad d = p_1\hat{y}_1 + p_2\hat{y}_2 \ .$$

For the case shown in the figure, the parameters p_1 and p_2 will be <u>positive</u>, so that d is also positive.

Call p_1 and p_2 the <u>shadow prices</u> of goods 1 and 2, respectfully, and for any output pair (y_1,y_2) (not necessarily on the line (1.3)), call the value of

$$p_1y_1 + p_2y_2$$

the <u>shadow value</u> of the output pair. It is intuitively clear from Figure 2 that <u>of all the feasible</u> output pairs, the given efficient pair $(\hat{y}_1,\hat{y}_2)$ has the largest shadow value.

Furthermore, Figure 2 makes it plausible that if one were to change the shadow prices and again search for the output pair that maximizes the shadow value, one would be led to another efficient pair. Indeed, by giving the shadow prices all possible non-negative values, one is led in turn to all the efficient output pairs.

Finally, suppose that the efficiency curve had the shape shown in Figure 3 (this could be the case for some production functions other than (1.1)). It would still be true that an output pair with maximum shadow value — for any given non-negative shadow prices — is efficient (e.g. the point y' in Figure 3), but there would now be efficient output pairs (e.g. the point y") that could not be obtained by maximizing shadow value — for any non-negative prices.

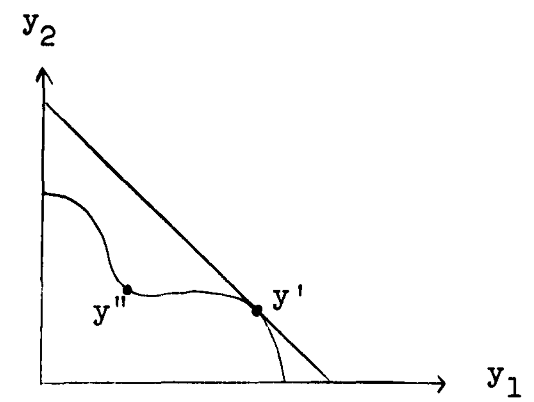


Figure 3

Shadow prices can also be used to characterize output pairs that are optimal in the sense of maximizing a social welfare function, provided the function has a suitable form. For example, suppose that the welfare function has the form

$$u(y_1,y_2) = \omega_1 \log y_1 + \omega_2 \log y_2 , \tag{1.5}$$

where ω_1 and ω_2 are given positive numbers (see Chapter II, Section 3). A typical iso-welfare curve is shown in Figure 4.

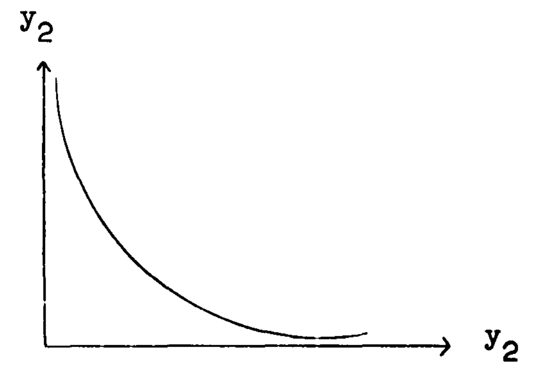


Figure 4

It is intuitively clear (see Figure 5) that the feasible output pair with maximum welfare is the point $\hat{y}$ at which an iso-welfare curve is tangent to the efficiency curve.

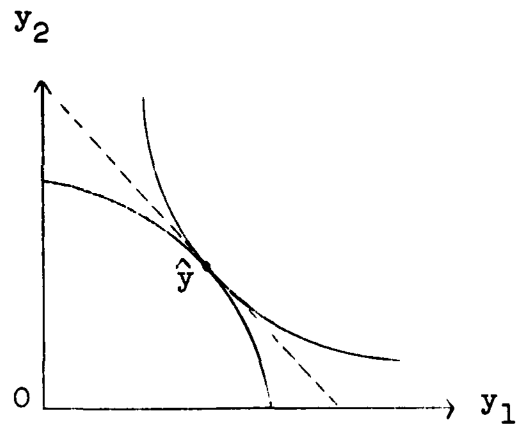


Figure 5

The dotted line that is tangent to both the efficiency curve and the iso-welfare curve at the optimal point $\hat{y}$ determines again shadow prices such that $\hat{y}$ has maximum value among all feasible output pairs.

The particular configuration shown in Figure 5 is possible because the production and welfare functions have certain special properties. In particular, the production function (1.1) exhibits decreasing marginal productivity of the inputs, and the welfare function (1.5) exhibits decreasing marginal welfare for each good.

Since the "price line" is tangent to the iso-welfare curve as well as to the efficiency curve at the point $\hat{y}$, it can be shown that one can take as shadow prices any pair of numbers proportional to the partial derivatives of the welfare function, evaluated at $\hat{y}$. Since in the present example

$$\frac{\partial u(y_1, y_2)}{\partial y_1} = \frac{\omega_1}{y_1}$$

one can take

$$p_1 = \frac{\omega_1}{\hat{y}_1} \quad . \tag{1.6}$$

Thus p_1 can be interpreted as the increase in welfare that would be achieved were one to be able to increase the optimal amount of good i by one unit (without decreasing the amount of the other good).

Figure 5 also illustrates part (a) of result (2) of Section 1.1, namely, that the output pair $(\hat{y}_1,\hat{y}_2)$ gives the largest welfare of all the points on the "price line" (or even below the price line). Since all the points on the price line have the same shadow value (and, of all the points below, a smaller value), the pair $(\hat{y}_1,\hat{y}_2)$ is the best among all those that involve no larger "expenditure" of units of shadow value.

To illustrate part (b) of the results of types (2) and (3) (see Section 1.1) requires the explicit calculation of an optimal output pair, which I will present using the linear-logarithmic welfare function (1.5).

Taking the logarithm of the production functions (1.1), and substituting $(K-K_1)$ for K_2 and $(L-L_1)$ for L_2, one obtains

$$(1.7)\qquad \begin{aligned} \log y_1 &= \beta_1 + \alpha_1 \log K_1 + (1-\alpha_1) \log L_1 , \\ \log y_2 &= \beta_2 + \alpha_2 \log(K-K_1) + (1-\alpha_2) \log(L-L_1) . \end{aligned}$$

The variables to be chosen are K_1 and L_1. Using the welfare function (1.5), the welfare v obtained for any choice of K_1 and L_1 is

$$(1.8)\qquad \begin{aligned} v = {} & \omega_1[\beta_1 + \alpha_1 \log K_1 + (1-\alpha_1) \log L_1] \\ & + \omega_2[\beta_2 + \alpha_2 \log (K-K_1) + (1-\alpha_2) \log (L-L_1)]. \end{aligned}$$

(It is assumed that $0 < K_1 < K$, $0 < L_1 < L$.) The optimum is obtained by setting the partial derivatives of v with respect to K_1 and L_1 equal to zero.

$$\frac{\partial v}{\partial K_1} = \frac{\omega_1\alpha_1}{K_1} - \frac{\omega_2\alpha_2}{K-K_1} = 0 \ ,$$

(1.9)

$$\frac{\partial v}{\partial L_1} = \frac{\omega_1(1-\alpha_1)}{L_1} - \frac{\omega_2(k-\alpha_2)}{L-L_1} = 0 \ .$$

This is easily solved to give

$$\hat{K}_1 = \left[\frac{\omega_1\alpha_1}{\omega_1\alpha_1 + \omega_2\alpha_2}\right] K \ ,$$

(1.10)

$$\hat{L}_1 = \left[\frac{\omega_1(1-\alpha_1)}{\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2)}\right] L \ .$$

Let p_1 and p_2 denote the shadow prices of goods 1 and 2 respectively, as before, and let r and s denote the shadow prices of capital and labor; then the <u>shadow profit</u> derived from the production of good i is

(1.11) $\quad p_i y_i - rK_i - sL_i \ .$

Suppose that the prices p_i are given by (1.6), and the prices r and s by

(1.12) $\quad r = \dfrac{\omega_1\alpha_1 + \omega_2\alpha_2}{K} \ , \qquad s = \dfrac{\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2)}{L} \ .$

I will show that, with shadow prices so defined,

(a) the optimal allocations (1.10) yield zero shadow profits, and

(b) any allocations that are not proportional to the optimal ones yield negative shadow profits.

In particular, then, I will have shown that the optimal allocations maximize shadow profits in each production sector.

To demonstrate (a) above, substitute the given values of the shadow prices, (1.6) and (1.12), and the optimal values of K_1 and L_1, (1.10), in the expression for profit, (1.11); this gives

$$\text{(1.13)} \qquad \text{profit} = \omega_1 - \omega_1\alpha_1 - \omega_1(1-\alpha_1) = 0.$$

Secondly, if (K_1, L_1) is not proportional to $(\hat{K}_1, \hat{L}_1)$, and y_1 denotes the resulting output, then profit is

$$\text{(1.14)} \qquad \omega_1\left(\frac{y_1}{\hat{y}_1}\right) - \omega_1\alpha_1\left(\frac{K_1}{\hat{K}_1}\right) - \omega_1(1-\alpha_1)\left(\frac{L_1}{\hat{L}_1}\right) .$$

To show that the profit (1.14) is negative, it suffices to show

$$\text{(1.15)} \qquad \frac{y_1}{\hat{y}_1} < \alpha_1\left(\frac{K_1}{\hat{K}_1}\right) + (1-\alpha_1)\left(\frac{L_1}{\hat{L}_1}\right) .$$

But from (1.7)

$$\left(\frac{y_1}{\hat{y}_1}\right) = \left(\frac{K_1}{\hat{K}_1}\right)^{\alpha_1}\left(\frac{L_1}{\hat{L}_1}\right)^{1-\alpha_1} < \alpha_1\left(\frac{K_1}{\hat{K}_1}\right) + (1-\alpha_1)\left(\frac{L_1}{\hat{L}_1}\right),$$

which proves (1.15). (The last inequality follows from the well-known fact that a weighted geometric mean is smaller than the corresponding weighted arithmetic mean.)

One may give an interpretation to the shadow prices r and s similar to that given to p_1 and p_2. First rewrite the expression (1.8) for welfare directly in terms of K_1, K_2, L_1 and L_2:

$$\text{(1.16)} \qquad v = \omega_1[\beta_1 + \alpha_1 \log K_1 + (1-\alpha_1) \log L_1]$$
$$+ \omega_2[\beta_2 + \alpha_2 \log K_2 + (1-\alpha_2) \log L_2].$$

The effect on v of a small change in K_1 (holding the other inputs constant) is given by the partial derivative

$$(1.17) \qquad \frac{\partial v}{\partial K_1} = \frac{\omega_1 \alpha_1}{K_1} .$$

Substituting the optimal value (1.10) for K_1 in (1.17) gives

$$(1.18a) \qquad \left.\frac{\partial v}{\partial K_1}\right|_{K_1=\hat{K}_1} = \frac{\omega_1\alpha_1 + \omega_2\alpha_2}{K} = r .$$

Similarly, one easily verifies that

$$(1.18b) \qquad \left.\frac{\partial v}{\partial K_2}\right|_{K_2=\hat{K}_2} = r$$

$$(1.19) \qquad \left.\frac{\partial v}{\partial L_1}\right|_{L_1=\hat{L}_1} = \left.\frac{\partial v}{\partial L_2}\right|_{L_2=\hat{L}_2} = s .$$

Note that in (1.18) (in contrast to (1.9)), an increase in K_1 is <u>not</u> compensated by a corresponding decrease in K_2. Hence r can be interpreted as the <u>increase in welfare that would be achieved if the total stock</u> K <u>of capital were increased by one unit</u>. A similar interpretation holds for s.

1.3 Shadow Prices, Present Value, and Optimality

Let $p_i(t)$ denote the <u>shadow price</u> of commodity i in period t. If one makes an analogy with a market economy, then $p_i(t)$ is to be interpreted as the <u>present value</u> or discounted value of one unit of commodity i made available, or used, at the beginning of period t. The present value of a commodity vector c(t) is therefore

$$(1.20) \qquad \sum_{i=1}^{M} p_i(t)c_i(t) ,$$

and the present value of a sequence c(1), ..., c(T) is

$$(1.21) \qquad \sum_{t=1}^{T} \sum_{i=1}^{M} p_i(t)c_i(t) .$$

It may be helpful to say a little about the relation between formula (1.21) and the usual way of computing present value from market prices and interest rates. Let $p_i^*(t)$ denote the market price of commodity i that is current at the beginning of period t, and let there be a single rate of interest r. Then the present value of the commodity vector c(t) would be

$$(1.22) \qquad \frac{1}{(1+r)^{t-1}} \sum_i p_i^*(t) c_i(t),$$

and the present value of the sequence c(1), c(2), ..., etc. would be

$$(1.23) \qquad \sum_t \frac{1}{(1+r)^{t-1}} \sum_i p_i^*(t) c_i(t).$$

More generally, there would be a different rate of interest for each interval of time from 1 to t (e.g. "long" and "short" term interest rates). Let r_t be the interest rate (per period) for the time interval from the beginning of period 1 to the beginning of period t; then the present value of sequence c(1), c(2), ..., etc. would be (compare with (1.23))

$$(1.24) \qquad \sum_t \frac{1}{(1+r_t)^{t-1}} \sum_i p_i^*(t) c_i(t) = \sum_t \sum_i \frac{p_i^*(t)}{(1+r_t)^{t-1}} c_i(t) .$$

By comparing (1.24) with (1.21) one sees that the shadow price $p_i(t)$ corresponds to the discounted market price

$$(1.25) \qquad \frac{p_i^*(t)}{(1+r_t)^{t-1}} .$$

It is important to note that the numbers (1.25) do <u>not</u> uniquely determine the sequence of market prices and rates of interest. Hence it is not possible to associate a unique sequence of shadow rates of interest with a given sequence of shadow prices $p_i(t)$.

Returning now to the shadow prices, one defines the __shadow profit__ associated with the input-output pair $[x(t),y(t)]$ by

$$(1.26) \qquad \sum_i p_i(t+1)y_i(t) - \sum_i p_i(t)x_i(t).$$

Notice that the output $y_i(t)$ is evaluated using the price $p_i(t+1)$, because that output is made available only at the beginning of period $(t+1)$. Summing (1.26) over t, one gets the total present value of the profit associated with the entire production program $[x(t),y(t)]$:

$$(1.27) \qquad \sum_t [\sum_i p_i(t+1)y_i(t) - \sum_i p_i(t)x_i(t)] \ .$$

If production is divided into "sectors", then there will be an input-output pair $[x^{(k)}(t),y^{(k)}(t)]$ for each sector k, with

$$(1.28) \qquad x(t) = \sum_k x^{(k)}(t) \ ,$$

$$(1.29) \qquad y(t) = \sum_k y^{(k)}(t) \ .$$

One can then calculate the shadow profit in each sector by formulas analogous to (1.26) and (1.27).

Before giving precise statements of the theorems on shadow prices and optimality, I must introduce some ideas concerning consumption and production possibilities. First, there may be constraints on consumption other than those imposed by limitations on productivity and on the availability of natural resources. For example, there may be some minimum consumption standards, derived from either biological or political considerations. Or it may be considered unacceptable for consumption to decrease at any time. The set of consumption sequences that are considered __acceptable a priori__ will be denoted by $\mathcal{A}$. This set will typically differ from the set of __feasible__ consumption sequences. The set of consumption sequences that are __both__ acceptable and feasible will be denoted by $\mathcal{C}$. The set $\mathcal{C}$ then, depends upon the set $\mathcal{A}$ of acceptable consumption sequences, upon the set $\mathcal{T}$ of feasible input-output pairs, and on the sequence of natural resources $q(t)$.

In the following statement of theorems on shadow prices and optimality, I use the concepts of <u>convexity</u> and <u>concavity</u> as applied to certain sets and functions.[+] Roughly speaking,

(a) If the production possibility set $\mathcal{T}$ is convex, then (globally) production has the properties of non-increasing returns to scale and non-increasing marginal productivity of the various factors.

(b) If the welfare function U is concave, then each commodity at each period has non-increasing marginal welfare.

Let $\mathcal{C}$ denote the set of consumption sequences $\{c(t)\}$ that correspond to the set of all feasible programs. It can be shown that if $\mathcal{T}$ is convex, then so is $\mathcal{C}$.

I present the following theorems without proof.

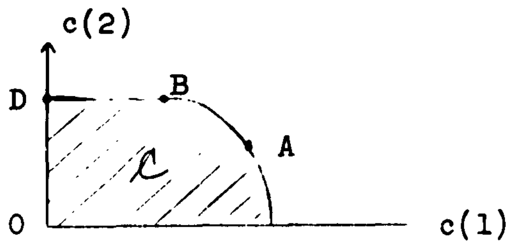
(1) <u>A "best" program has a maximum present value</u>; i.e.,

(1a) If for some <u>positive</u> shadow prices $p_i(t)$ a consumption sequence has maximum present value in the set $\mathcal{C}$, then it is efficient in $\mathcal{C}$.

(1b) Assume that $\mathcal{T}$ is convex. If a consumption sequence is efficient, in $\mathcal{C}$, then for some suitably chosen <u>non-negative</u> shadow prices, it has maximum present value in $\mathcal{C}$.

(1c) Assume that $\mathcal{T}$ is convex, and that the welfare function U is concave and continuous. If the consumption sequence maximizes U on $\mathcal{C}$, then for some suitably chosen shadow prices it maximizes present value on $\mathcal{C}$.

One aspect of theorems (1a) and (1b) is illustrated in the following figure.



[+] For an introduction to these concepts, see BERGE, Chapter VIII; or EGGLESTON; or HADLEY (on convex sets).

The figure represents a situation in which there is one commodity and two periods. The shaded region represents the set $\mathcal{C}$. Point A is an efficient consumption pair for which the corresponding shadow prices are positive. Point B is an efficient pair for which the shadow price of consumption in period 1 is zero. Point D maximizes present value on $\mathcal{C}$ if the shadow price of consumption in period 1 is zero, but D is not efficient since point B provides more consumption in period 1, and the same in period 2.

Other aspects of (1a)-(1c) are illustrated by Figures 2-5 of Section 1.2, with suitable relabelling of the axes.

(2) <u>A "best" program produces the most welfare for the given "shadow expenditure", and the most shadow profit from production</u>.

Assume that $\mathcal{J}$ is convex, that U is strictly+ concave and continuous, and that there is a conceivable non-negative consumption sequence that is better than the best feasible consumption sequence. A consumption sequence $\hat{c}(t)$ is best in $\mathcal{C}$ if and only if for some suitably chosen shadow prices

(a) $\hat{c}(t)$ is a best consumption sequence among all those non-negative sequences whose present value is no greater than that of $\hat{c}(t)$;

(b) the sequence of input-output pairs $[\hat{x}(t),\hat{y}(t)]$ associated with $\hat{c}(t)$ has maximum shadow profit among all feasible programs.

To the above assumptions must be added the proviso that the present value of the sequence $\hat{c}(t)$ is not the minimum possible in the set $\mathcal{C}$.

(3) <u>Decentralization of the profit calculation</u>.

Suppose that the production possibility set $\mathcal{J}$ is what we shall call a <u>sum of sets</u>

$$\mathcal{J} = \sum_{k=1}^{K} \mathcal{J}_k \, ;$$

+Roughly speaking, strictly decreasing marginal utility.

i.e., suppose that an input-output pair (x,y) is feasible if and only if there is an input-output pair $(x^{(k)},y^{(k)})$ in $\mathcal{T}_k$, for each k, such that

$$(x,y) = \sum_k (x^{(k)},y^{(k)}).$$

This assumption expresses what is sometimes called the <u>absence of external economies or diseconomies among sectors</u>. If this assumption is added to those of theorem (2) above, and if each set $\mathcal{T}_k$ is convex, then conclusion (b) of result (2) holds for each of the K sectors separately.

In the case of planning for an infinite number of time periods, similar results can be obtained, but additional assumptions are needed (see DEBREU, 1954; MALINVAUD, 1953 and 1961a).

In some situations, not every non-negative consumption sequence may be considered acceptable from the point of view of the planner. In other words, criteria other than the welfare function may be brought into the planning problem, in the form of constraints on the set of consumption sequences from which a choice is to be made. For example, there may be minimum requirements for certain commodities (food, housing) and maximum limits on others (leisure). If the set of acceptable consumption sequences is convex, and if the set $\mathcal{C}$ above is redefined to be the set of consumption sequences that are both feasible and acceptable, then Theorems (1)-(3) above remain correct as stated, except for the following change: in Theorem (2) one must assume that there is an <u>acceptable</u> consumption sequence that is better than the best feasible and acceptable consumption sequence.

1.4 Some Remarks on the Uses of Shadow Prices in Planning

The example given in Section 1.3 might give the impression that shadow prices are useless for economic planning, since it would appear that in order to calculate the correct shadow prices corresponding to the solution of a planning problem, it is necessary to calculate the solution of the problem!

Actually, the situation need not be as bad as this first impression might indicate. First, the problem of finding the appropriate prices might be easier computationally than the original problem. This sometimes occurs in the case in which the problem of finding an optimal program reduces to a linear programming problem (as in the case of linear activity analysis). Here, the prices are the so-called dual variables, and the dual problem may be easier than the primal problem.

Secondly, computational schemes have been proposed in which one successively adjusts the economic program, then the shadow prices, then the program again, etc., with convergence towards the optimal program and the correct prices [see ARROW and HURWICZ (1957)(1960)]. In particular, in the case of several production sectors, an iterative process that takes advantage of the "decomposition" of the production set can achieve considerable reduction in computation, or suggest ways of decentralizing the computation [see DANTZIG and WOLFE (1960), MALINVAUD (1961b), and again ARROW and HURWICZ (1960)]. Thirdly, application of the shadow price theorems may yield theoretical insights into the structure of optimal programs.

I will not have the time to discuss points one and two on computation, although they are important and interesting; I do intend to return to point three later in these lectures.

Finally, I should point out one danger in the use — or rather misuse — of shadow prices. If one cannot solve the computational problem of determining an optimal program, one may be tempted to guess at proper shadow prices and proceed from there. In particular, it is tempting to use observed market prices for this guess. Of course, the market prices, when used as shadow prices, need not lead to a feasible program; if they do, that program may be far from optimal from the social point of view. The use of shadow prices does not release the planner from the social responsibility of formulating fairly definite criteria of social welfare.

When market prices are used as a basis for estimating correct shadow prices, the hardest thing to determine is typically the appropriate rate of interest (or rates of interest). Thus there is typically much controversy on what rate of interest to use in planning public investment in roads, hydroelectric plants, etc. This is a backwards way to attack the programming problem; a more sensible way is to determine a feasible program with a desirable (if not optimal) consumption sequence, and then see whether there is some set of interest rates (and other shadow prices) that rationalizes the program in the sense of the above theorems. If not, the program can be adjusted, perhaps using some of the iterative techniques described in the above-mentioned references, and the process of testing the program can be repeated. (Some processes of this kind would seem to be a feature of current French planning; see MASSÉ.)

The problem is somewhat different if one is trying to choose not an overall program but a change in, or addition to, some already determined overall program. An example would be the choice of the best scale or location of a hydroelectric plant in a country that already has an overall economic plan. In this case it would be reasonable to use the shadow prices (and, in particular, the interest rates) that had already been used in the determination of the rest of the plan, provided the plan as a whole was considered approximately optimal. I shall not, however, go into this class of problems in these notes.

2. The Rate of Return

Although the criterion of rate of return has been much discussed, and often advocated, its usefulness is questionable. On the one hand, in those cases in which the criterion of rate of return gives correct results, the criterion of present value also gives correct results, and is just as easy, if not easier, to apply. On the other hand, the criterion can lead to incorrect results, even under circumstances in which the criterion of present value works well. Nevertheless, because of the widespread use of the rate of return, some discussion of it seems desirable.

This criterion is typically used for choosing among individual investments, private or public, rather than among entire national programs. However, in principle it can be applied as well to national programs, provided one already has a numerical measure of "income" in each period.

Suppose that one is considering a sequence of incomes, v_1, v_2, ..., etc. These may be total incomes (in each period), or they may be increments of income associated with a particular investment project under consideration. In the latter case, the initial income (or incomes) in the sequence will typically be negative, and the later incomes will typically be positive. Before defining the rate of return in a general way, it may be well to give two simple examples.

Example 1. $v_1 = -K, \quad v_2 = K + v, \quad v_t = 0 \quad \text{for } t \geq 3.$

In this example, one invests K in period 1, gets back K + v in period 2, and that's the end of it. It is not unnatural here to call the quantity (v/K) the rate of return.

Example 2. $v_1 = -K, \quad v_t = v \quad \text{for } t \geq 2.$

In this second example one invests K in the first period, and gets back v in every following period, ad infinitum. Such a situation could arise if the investment opportunity of Example 1 were available in every period. Again, it is not unnatural to call (v/K) the "rate of return".

In Example 1, suppose that the rate of interest were r; then the present value of the income received in period 2 would be

$$(2.1)\qquad \frac{K + v}{1 + r}\ .$$

Suppose further than $r = v/K$; then the present value (2.1) would exactly equal K. In other words, the quantity (v/K) is exactly that interest rate that makes the present value of future income equal to the initial cost.

In Example 2, for a given rate of interest r, the present value of future income ($t \geq 2$) is

$$(2.2)\qquad \sum_{t=2}^{\infty} \frac{v}{(1+r)^{t-1}} = \left(\frac{v}{1+r}\right) \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^{t}$$
$$= \left(\frac{v}{1+r}\right) \frac{1}{1 - \frac{1}{1+r}} = v/r\ .$$

Hence if $r = (v/K)$, then again the present value of future income equals the initial cost.

More generally, for any sequence of incomes v_1, v_2, ..., etc., the <u>rate of return</u> is defined as that rate of interest that would make the present value of the income sequence equal zero. Formally, given the sequence $v(t)$, define the present value

$$(2.3)\qquad \phi(r) = \sum_{t=1}^{\infty} \frac{v_t}{(1+r)^{t-1}}\ .$$

The rate of return $\tilde{r}$ is defined as the real-valued solution of

$$(2.4)\qquad \phi(\tilde{r}) = 0\ .$$

It should be pointed out that, in general, (2.4) may have <u>no solution</u>, or may have <u>several</u> solutions, so that the rate of return is not really defined for the entire class of all possible income sequences. As a further restriction, a value for the rate of return is usually not considered sensible unless it is > -1. This is because one usually thinks of the discount factor $(1/1+r)$ as being > 0. In particular, if <u>all</u> of the v_t are non-negative, and some positive, including v_1, then the discount factor would have to be negative, if anything! (E.g., if $v_1 = v_2 = 1$, and $v_t = 0$ for $t \geq 3$, then $\tilde{r} = 2$.)

It is often thought that the higher the rate of return, the better is the income sequence. However, it is easy to construct examples in which there is no monotonic relation between rate of return and present value calculated at a given shadow interest rate (see MASSÉ, pp. 23-24). In view of the intimate relation between optimality and maximum present value (see Section 1 of this chapter), this shows that maximizing the rate of return can not be guaranteed to lead to optimal programs. A suitable example can be constructed along the lines of the example at the end of Section 3 of this chapter.

There are, however, three somewhat interesting results connecting the rate of return and optimality. Let r_o be a given rate of interest, and call a program maximal if it yields an income sequence that has maximum present value among the alternative programs. I shall call the income sequence for the maximal program maximal, too. The three results are as follows:

(1) For "small" departures from a maximal program, the rate of return on the marginal income sequence (the sequence of increments) equals the given interest rate r_o.

(2) If $r_o > 0$, and all incomes are non-negative after the first period, then for sequences whose rates of return are sufficiently close to r_o, those sequences with the higher rate of return will also have larger present value. In particular, if the maximal sequence has a rate of return equal to r_o, then r_o is a (local) maximum of the rate of return.

(3) If the set of alternative income sequences (i.e., the sequences corresponding to alternative programs) exhibits constant returns to scale, then a maximal program also has a maximum rate of return.

To prove the above propositions, suppose that the set of alternative income sequences (or programs) is indexed by a parameter θ. Modifying slightly the notation of (2.3), let $\phi(r,\theta)$ denote the present value of a sequence $\{v_t(\theta)\}$, if the rate of interest is r, thus

$$\phi(r,\theta) = \sum_{t=1}^{\infty} \frac{v_t(\theta)}{(1+r)^{t-1}} .$$

Also, let ϕ_r and ϕ_θ denote the partial derivatives of ϕ with respect to r and θ_1 respectively.

Let r_o be a fixed interest rate, and let $\hat{\theta}$ be a maximal program; then

(2.6) $$\phi_\theta(r_o,\hat{\theta}) = 0 ,$$

or, from (2.5),

(2.7) $$\sum_{t=1}^{\infty} \frac{v_t'(\hat{\theta})}{(1+r_o)^{t-1}} = 0$$

(assuming all the necessary properties of differentiability and convergence). Hence r_o is a rate of return for the sequence $\{v_t'(\hat{\theta})\}$, which proves result (1).

For any θ, suppose that $\tilde{r}(\theta)$ is the corresponding rate of return; the function $\tilde{r}(\theta)$ is defined implicitly by

(2.8) $$\phi[\tilde{r}(\theta),\theta] = 0 .$$

Differentiating (2.8) with respect to θ gives

(2.9) $$\phi_r[\tilde{r}(\theta),\theta]\tilde{r}'(\theta) + \phi_\theta[\tilde{r}(\theta),\theta] = 0 .$$

Solving for $\tilde{r}'(\theta)$:

(2.10) $$\tilde{r}'(\theta) = -\frac{\phi_\theta[\tilde{r}(\theta),\theta]}{\phi_r[\tilde{r}(\theta),\theta]} .$$

Under the assumptions of result (2), ϕ_r is negative; hence $\tilde{r}'(\theta)$ and $\phi_\theta[\tilde{r}(\theta),\theta]$ have the same sign. If $\theta \neq \hat{\theta}$, and if $\tilde{r}(\theta)$ is sufficiently close to $\tilde{r}_o$, then $\phi_\theta[\tilde{r}(\theta),\theta]$ and $\phi_\theta[r_o,\theta]$ have the same sign, and hence so do $\tilde{r}'(\theta)$ and $\phi_\theta[r_o,\theta]$, which proves the first part of result (2). Furthermore, $\tilde{r}(\hat{\theta}) = r_o$, one has

$$\phi_\theta[\tilde{r}(\hat{\theta}),\hat{\theta}] = 0 ,$$

so that, by (2.10), $\hat{r}'(\hat{\theta}) = 0$, and $\tilde{r}(\theta)$ has a local maximum at $\hat{\theta}$, which proves the second part of result (2).

If there are constant returns to scale, i.e., if the availability of the sequence $\{v_t\}$ implies the availability of the sequence $\{kv_t\}$ for all $k \geqq 0$, then any maximal program must have present value zero (if a maximum exists at all). But in that case the rate of return for the maximal program is equal to the interest rate, so that result (2) applies.

3. The Benefit-Cost Ratio

Another criterion that is widely discussed, but unreliable, is the so-called benefit-cost ratio. For any given sequence of incomes v_t, let B denote the present value of the positive incomes in the sequence, and (-K) the present value of the negative incomes; B is called the present value of benefits, and K the present value of costs. The benefit-cost ratio is

(3.1) $\quad R = B/K .$

In terms of the present notation, the (net) present value of the sequence $\{v_t\}$ is of course

(3.2) $\quad P = B - K .$

Figure 6 shows the lines of constant benefit-cost ratio in the (K,B) plane, and Figure 7 shows lines of constant present value.

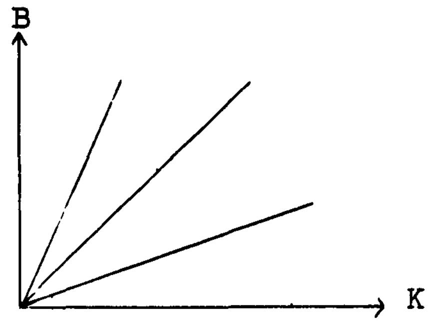


Figure 6

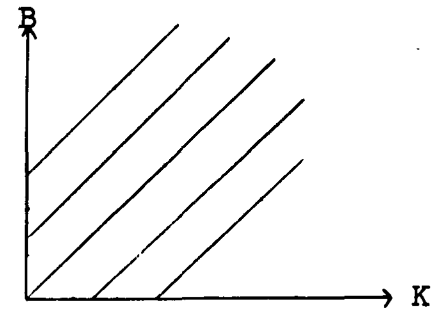


Figure 7

The proposition that maximizing the benefit-cost ratio typically does not lead to maximizing present value is, of course, equivalent to the proposition that minimizing average cost typically does not lead to maximum profit, and therefore probably needs no explanation here. Figure 8 will help the reader to recall the essential difficulty. In the figure, the shaded area represents the feasible cost-benefit pairs. Point M is the feasible point with maximum benefit-cost ratio, whereas point N has maximum present value. As one moves along the boundary of the feasible set from point L to point M, both R and P are increasing. Between M and N, R is falling, but P is rising. Beyond N, both R and P are falling.

By constructing figures similar to Figure 8, for Examples 1 and 2 of Section 2 of this chapter, one can easily see how maximizing the rate of return can lead to results that differ from maximizing present value.

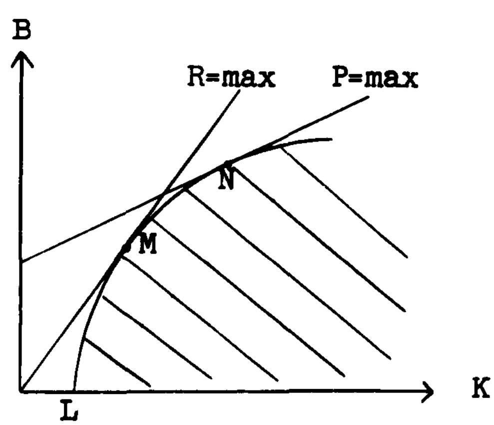


Figure 8

REFERENCES FOR CHAPTER III

K. J. ARROW and L. HURWICZ, "Decentralization and computation in resource allocation," Essays in Economics and Econometrics, Chapel Hill, N.C.: University of N.C. Press, 1960, 34-104.

C. BERGE, Espaces Topologiques, Paris: Dunod, 1959.

G. B. DANTZIG and P. WOLFE, "The decomposition principle for linear programs," Operations Research, 8 (1960), 101-111.

G. DEBREU, Theory of Value, New York: Wiley, 1959. (Cowles Foundation Monograph 17)

G. DEBREU, "Valuation equilibrium and Pareto optimum," Proceedings of the National Academy of Sciences of the U.S.A., 40 (1954), 588-592.

H. G. EGGLESTON, Convexity, Cambridge: The Cambridge University Press, 1958.

G. HADLEY, Linear Algebra, Reading, Mass.: Addison-Wesley, 1961.

T. C. KOOPMANS, Three Essays on the State of Economic Science, Chapter I, New York: McGraw-Hill, 1957.

E. MALINVAUD, "Capital accumulation and efficient allocation of resources," Econometrica, 21 (1953), 233-268.

E. MALINVAUD, "Efficient capital accumulation: a corrigendum," Working Paper No. 28, Management Science Research Center, University of California, Berkeley, June, 1961.

E. MALINVAUD, "L'intérêt comme indicateur de choix dans l'allocation des ressources," Association Internationale de Science Economique, Colloque sur la Théorie de l'Intérêt, Royaumont, Mars, 1962 (mimeographed).

E. MALINVAUD, "On decentralization in national planning," Working Paper No. 36, Management Science Research Center, University of California, Berkeley, August, 1961.

E. MALINVAUD, "The analogy between atemporal and intertemporal theories of resource allocation," Review of Economic Studies, 28 (1961).

P. MASSÉ, Le Choix des Investissements, Paris: Dunod, 1959.

CHAPTER IV

CALCULATION OF OPTIMAL PROGRAMS: AN EXAMPLE

1. Introduction

In this chapter I will bring together various elements introduced in the first three chapters, in the context of an extended example. For this example I will derive explicit formulas for the optimal programs, both for finite and infinite horizons, and show how certain aspects of these programs – e.g. the saving and consumption coefficients and the rates of growth – depend upon the various parameters of the production and social welfare functions. I will also investigate the corresponding shadow prices.

For this example, I will assume:

(1) There are two types of commodities, produced and non-produced (primary resources). The production and aging of the produced commodities follows the linear-logarithmic model of Chapter I, Section 3. The primary resources are made available exogenously, i.e. in a sequence of quantities that does not depend upon which program is adopted.

(2) Social welfare is a sum of discounted one-period welfares, and the one-period welfare function is linear-logarithmic, as in Chapter II, Section 3, Example (ii).

An important division of cases arises according to whether or not primary resources (in the above sense) actually do enter into the economy. In the special case of the absence of primary resources, the optimal programs approach, in the long run, proportional growth paths, i.e. paths in which the consumption, production, and stocks of all commodities grow exponentially at the same rate. This common growth factor is a product of two factors. The first factor is a weighted geometric mean of the savings factors for the various commodities. The second factor is the growth factor for production that would be achieved if there were no consumption, but goods and services were allocated to production in the same proportions as in the optimal program. This latter hypothetical growth factor is in turn typically less than the maximum that is technologically possible, because the "direction" of growth in the optimal program is pulled away from the direction of technologically maximum growth by the tastes for current consumption. In the limiting case of a social time discount factor of unity (no discounting of future welfare), consumption is zero, and production grows at the maximum rate that is technologically feasible.

If primary resources <u>are</u> present (and necessary for production), then the long-run behavior of output is determined by the long-run pattern of the sequence of primary resources. In particular, if a constant quantity of each primary resource is available each period, then output and consumption approach constants in the long run; whereas, if the supplies of all primary resources grow at the same rate, then asymptotically output and consumption also grow at that rate. Even if the social time discount factor is unity, consumption will typically be positive; indeed, the optimal program in this case is the program that yields the highest long-run level of welfare.

(In interpreting these results, it should be borne in mind that "consumption" here consists of all quantities used up that do not enter the productive process.)

Shadow prices can be calculated for the optimal programs in all of these cases; furthermore, in the case of long-run proportional growth, there will be a natural way of defining the asymptotic shadow interest rate. In all cases, the rate of interest exceeds, or equals, the rate of growth, according as the social time discount factor is less than, or equal to, unity.

I use the method of dynamic programming to derive these results. The essential idea of this method is to determine the maximum welfare achievable at any time as a function of the current stocks of commodities and of the number of periods remaining in the program.

In order to make the exposition easier to follow, and to bring out more clearly the various aspects of the problem, I give first an example with a single commodity, followed by an example with two commodities, one produced and the other a primary resource.

The general multicommodity case is then discussed, with a division into the two cases of absence and presence of primary resources.

2. The Case of One Commodity

Suppose there is only one commodity, whose production obeys the relation

$$(2.1)\qquad y = e^{\beta}x, \quad \text{or} \qquad \log y = \beta + \log x,$$

where x is the input, y is the output, and β is a given parameter. If consumption in period t is denoted by c(t), then the social welfare for the (T+1) periods 0, ..., T is assumed to be[+]

$$(2.2)\qquad v = \sum_{t=0}^{T} \delta^{t} \log c(t),$$

where δ is a positive parameter, the social time discount factor.

In each period t, one must decide how to divide the beginning-of-period stock z(t) into consumption c(t) and input x(t). Thus one has

$$(2.3)\qquad \begin{aligned} & c(t) + x(t) \leqq z(t)\ , && t = 0, \ldots, T\ ; \\ & c(t) \geqq 0,\ x(t) \geqq 0\ , && \\ & z(t+1) = e^{\beta}x(t)\ , && t = 0, \ldots, T-1\ . \end{aligned}$$

The problem is to maximize welfare v, as given by (2.2) subject to (2.3), and given z(0) and T.

[+]For convenience, I start counting time at t = 0 here, instead of t = 1.

The Dynamic Programming Valuation Function

The problem will be solved by determining, recursively, a function $G_T(z)$ that gives the maximum possible welfare for a given initial stock $z(0) = z$ in a program $(T+1)$ periods long.[+]

Consider first the case $T = 0$. The best program is clearly the one in which all of the initial stock is consumed. Hence

$$G_0(z) = \log z \; . \tag{2.4}$$

Now consider the case $T = 1$. If $c(0)$ is consumed in period 0, the initial stock in period 1 will be, by (2.3),

$$z(1) = e^{\beta}[z(0) - c(0)] \;, \quad \text{or}$$

$$\log z(1) = \beta + \log[z(0) - c(0)] \; . \tag{2.5}$$

But since period 1 is the last period, all of $z(1)$ will be consumed, so that total welfare will be

$$v = \log c(0) + \delta \log z(1) = \log c(0) + \delta(\beta + \log[z(0)-c(0)]) . \tag{2.6}$$

To maximize v, set the derivative of v with respect to $c(0)$ equal to zero

$$\frac{1}{c(0)} - \frac{\delta}{z(0) - c(0)} = 0 \;,$$

which yields

$$c(0) = \left(\frac{1}{1+\delta}\right)z(0) \;, \tag{2.7}$$

$$x(0) = \left(\frac{1}{1+\delta}\right)z(0) \; .$$

[+] For the one-commodity case, a direct rather than recursive attack on the problem is probably simpler, but the aim here is to introduce the recursive method in a simple context.

The maximum value of welfare is therefore (substituting (2.7) in (2.6)

$$G_1[z(0)] = \log \left(\frac{1}{1+\delta}\right)z(0) + \delta[\beta + \log \left(\frac{\delta}{1+\delta}\right)z(0)] , \tag{2.8}$$

or, after some rearrangement,

$$G_1[z(0)] = (1+\delta) \log z(0) + \log \left(\frac{1}{1+\delta}\right) + \delta \log \left(\frac{\delta}{1+\delta}\right) + \delta\beta. \tag{2.9}$$

Note that from (2.6) one could have written

$$G_1[z(0)] = \max_{c(0)} [\log c(0) + \delta G_0(e^{\beta}[z(0) - c(0)])]. \tag{2.10}$$

Similarly, at the beginning of period 1 in a program with horizon T (i.e. a program with (T+1) periods) one faces a remaining program of T periods, but with an initial stock of $z(1)$, instead of $z(0)$. Hence G_T is related to G_{T-1} by

$$G_T[z(0)] = \max_{c(0)} [\log c(0) + \delta G_{T-1}(e^{\beta}[z(0) - c(0)])]. \tag{2.11}$$

I will shortly show that G_T is given by

$$G_T(z) = \left(\sum_{t=0}^{T} \delta^t\right) \log z + K_T , \tag{2.12}$$

where K_T is a quantity that depends upon T, but not upon z. The formula for K_T is given below, but is not important for the time being.

The significance of the particular form (2.12) of G_T is recalled by going back to the recursive relation (2.11). In order to determine what to do in the first period of a program with horizon T, we must maximize

$$\log c(0) + \delta G_{T-1}[z(1)] .$$

From (2.12), this is equal to

$$\log c(0) + \delta\left[\sum_{t=0}^{T-1} \delta^t\right]\log z(1) + K_{T-1}$$

$$= \log c(0) + \left(\sum_{t=1}^{T} \delta^t\right) \log z(1) + \delta K_{T-1} .$$

To maximize this last quantity, it is sufficient to maximize

$$\log c(0) + \left(\sum_{t=1}^{T} \delta^t\right) \log z(1) ,$$

since K_{T-1} is independent of $c(0)$ and $z(1)$. A comparison of this last quantity with (2.6) shows that this maximization problem is formally the same as that for the case $T = 1$, except that δ has been replaced by $\left(\sum_{t=1}^{T} \delta^t\right)$.

What is more, the problem of determining what to do in period t of a program with horizon T is equivalent to a problem of determining the <u>first</u> step of a program with horizon $(T-t)$, and according to the remarks just made, this latter problem has the same form as a problem with horizon 1. <u>Hence the problem of determining any single step of a program with arbitrary (finite) horizon can be transformed into an "equivalent" problem of determining the first step of a program with horizon</u> 1.

I now prove (2.12), and the proof will show, incidentally, that K_T is determined recursively by

$$K_T = \left(\sum_1^T \delta^t\right) \log\left(\sum_1^T \delta^t\right) - \left(\sum_0^T \delta^t\right) \log\left(\sum_0^T \delta^t\right) \tag{2.13}$$

$$+ \beta \sum_1^T \delta^t + \delta\, K_{T-1} ,$$

$$K_0 = 0 .$$

The proof is by induction. It has already been shown that (2.12) and (2.13) hold for T = 0. Suppose now that it holds for all horizons up to and including (T-1). Then from (2.11) one has

$$(2.14)\qquad G_T[z(0)] = \max_{c(0)}\left[\log c(0) + \delta\left(\sum_0^{T-1} \delta^t\right)(\beta + \log[z(0)-c(0)]) + \delta K_{T-1}\right] .$$

Performing the indicated maximization, one gets

$$\frac{1}{c(0)} - \frac{\delta \sum_0^{T-1} \delta^t}{z(0) - c(0)} = 0 ,$$

or

$$(2.15)\qquad \begin{cases} c(0) = \left(\dfrac{1}{\sum_0^T \delta^t}\right) z(0) , \\[2ex] x(0) = z(0) - c(0) = \left(\dfrac{\sum_1^T \delta^t}{\sum_0^T \delta^t}\right) z(0) . \end{cases}$$

Substituting (2.15) in (2.14), one easily verifies the desired results.

Consumption and Saving Coefficients

According to the principle already used in relating G_T to G_{T-1} (see (2.11)), one obtains immediately from (2.15) the optimal consumption and saving in period t for a program with horizon T, since one starts period t with a stock z(t), and there are (T-t) periods remaining ir the program after the current period t. Thus,

$$(2.16)\qquad \begin{aligned} c(t) &= \left(\frac{1}{\sum_{k=0}^{T-t} \delta^k} \right) z(t) , \\ x(t) &= \left(\sum_{k=1}^{T-t} \delta^k \Big/ \sum_{k=0}^{T-t} \delta^k \right) z(t) . \end{aligned}$$

Notice that the ratios of $c(t)$ and $x(t)$ to $z(t)$ depend upon t and T, but not on $z(t)$.

Infinite Programs

For the rest of this section, the discussion will be limited to the case of an infinite horizon. Letting $T \to \infty$ in (2.16), for fixed t, one obtains for the case $\delta < 1$

$$(2.17)\qquad \begin{aligned} c(t) &= (1-\delta)z(t) , \\ x(t) &= \delta z(t) . \end{aligned}$$

Thus one has a constant saving factor, equal to the social time discount factor in all periods.

Since the sum in (2.12) does not converge for $\delta = 1$, we can not strictly speak of the optimal infinite program for this case. Nevertheless, we see that as $\delta \to 1$, for fixed t, consumption approaches 0 and "saving" (input) approaches $z(t)$.

Returning to the case $\delta < 1$, one easily computes the evolution of output $z(t)$ in time as

$$z(t) = \delta e^{\beta} z(t-1) .$$

The solution of this difference equation is

$$(2.18)\qquad z(t) = (\delta e^{\beta})^t \, z(0) .$$

Hence output, and therefore consumption, grows exponentially, with growth factor δe^{β} (the growth rate is $\delta e^{\beta}-1$). The maximum growth rate for output is achieved if $\delta = 1$, but in this case there is no consumption! If $\delta' < \delta'' < 1$, and $\delta' e^{\beta} > 1$, then the optimal program corresponding to δ' will start with higher consumption than the optimal program corresponding to δ'', but eventually consumption in the second program will surpass that of the first, since it is growing exponentially at a larger rate.

The Interest Rate for an Infinite Program

There is only one commodity, so that the only shadow prices associated with an optimal program correspond to rates of interest. Indeed, since growth occurs at a constant rate, the shadow rate of interest must be constant. I will show that it equals $(e^{\beta}-1)$ for all optimal plans; this corresponds to a shadow discount factor of $e^{-\beta}$. In particular, the shadow rate of interest exceeds the rate of growth if $\delta < 1$.

To show that the rate of interest is $(e^{\beta}-1)$, it suffices to show that, using this rate (and this rate only), shadow profit is maximum for the production plan of the optimal program.

For a rate of interest r, the present value of profit for period t is

$$\frac{y(t)}{(1+r)^{t+1}} - \frac{x(t)}{(1+r)^{t}} ,$$

or, using the production relation (2.1),

$$\frac{e^{\beta}x(t)}{(1+r)^{t+1}} - \frac{x(t)}{(1+r)^{t}} = \frac{x(t)}{(1+r)^{t+1}} (e^{\beta}-1-r) . \tag{2.19}$$

Three cases arise:

(i) If $e^{\beta}-1-r > 0$, then profit could be made arbitrarily large by making $x(t)$ sufficiently large.

(ii) If $e^{\beta}-1-r < 0$, then maximum profit (zero) is achieved by making $x(t) = 0$, i.e. by not producing.

(iii) If $e^{\beta}-1-r = 0$, then profit is zero for all production plans.

Hence the only interest rate for which an optimal plan gives maximum profit is $r = e^{\beta}-1$ (case (iii)).

It is of some interest to note what happens when a linear welfare function is used instead of a logarithmic-linear one. Suppose that in place of (2.2) one has the welfare function

$$v = \sum_{t=0}^{T} \delta^{t} c(t). \qquad (2.20)$$

It is easy to show that two cases arise in classifying the optimal programs:

(i) If $\delta e^{\beta} < 1$, then all of the initial stock $z(0)$ is consumed in period 0, and consumption is zero for the rest of the program.

(ii) If $\delta e^{\beta} > 1$, then consumption is zero until the final period T, at which time the stock $z(T) = e^{T\beta} z(0)$ is consumed.

In the boundary case $\delta e^{\beta} = 1$, all programs are equally good (provided nothing is thrown away).

Even this simple example indicates how the use of a purely linear welfare function can possibly lead to extreme - and absurd - results.

Exercise 1.

Verify part (a) of result (2), Section 1.1, Chapter III, for optimal infinite programs. [Hint: take the shadow "budget" to be equal to $z(0)$, the value of the initial stock, and treat the constrained maximum problem in the usual way, ignoring the fact that there are an infinite number of "unknowns" $c(t)$.]

Exercise 2.

Find the shadow interest rates for optimum programs with finite horizon T.

3. The Case of Two Commodities, Produced and Primary

Suppose that there are only two commodities, of which the first is produced from inputs of both, according to a linear-logarithmic production function, and the second is a primary resource (see Section 1). More precisely, if in any period the inputs of the two commodities are x_1 and x_2, respectively, then the output of good 1 is

$$(3.1) \qquad y_1 = e^{\beta} x_1^{\alpha_1} x_2^{\alpha_2}, \qquad \text{or}$$

$$\log y_1 = \beta + \alpha_1 \log x_1 + \alpha_2 \log x_2 ,$$

where β, α_1, α_2 are given parameters, and α_1 and α_2 are positive. I shall also have occasion to use the assumption of constant returns to scale,

$$(3.2) \qquad \alpha_1 + \alpha_2 = 1.$$

As far as commodity 2 goes, a quantity $q(t)$ is exogenously made available at the beginning of each period t.

Suppose further that the social welfare function is given by

$$(3.3) \qquad v = \sum_{t=0}^{T} \delta^t [\omega_1 \log c_1(t) + \omega_2 \log c_2(t)],$$

where δ, ω_1, and ω_2 are positive parameters, with

$$(3.4) \qquad \omega_1 + \omega_2 = 1,$$

and $c_i(t)$ is, as usual, the consumption of commodity i in period t.

The programming problem is to choose the consumptions $c_i(t)$ and the inputs $x_i(t)$ to maximize the welfare (3.3) subject to (3.1) and

$$(3.5)\qquad \left.\begin{array}{l} c_i(t) + x_i(t) \leqq z_i(t) \\ c_i(t),\ x_i(t) \geqq 0 \end{array}\right\} \qquad \begin{array}{l} i = 1,\ 2, \\ t = 0,\ \ldots,\ T, \end{array}$$

$$z_1(t+1) = y_1(t) \qquad t = 1,\ \ldots,\ T,$$

$$z_2(t) = q(t) \qquad t = 1,\ \ldots,\ T,$$

where $z_1(0)$, $z_2(0)$, and $q(1), \ldots, q(T)$ are given.

Notice that since only one good is produced, there is still no problem of allocation of inputs among alternative uses, but only the problem of allocating the stocks $z_i(t)$ between consumption and production.

The Dynamic Programming Valuation Function

Again, I use the dynamic programming technique, determining recursively a function G_T that gives the maximum possible welfare for a given horizon T, a given initial stock $z(0) = \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$, and a given sequence of primary resources $q(1), \ldots, q(T)$.

For T = 0, all initial stocks are consumed, so that

$$(3.6)\qquad G_0[z(0)] = \omega_1 \log z_1(0) + \omega_2 \log z_2(0).$$

Now consider T = 1. The welfare is

$$(3.7)\qquad v = \omega_1 \log c_1(0) + \omega_2 \log c_2(0)$$
$$+ \delta[\omega_1[\beta\alpha_1 \log x_1(0)\ \alpha_2 \log x_2(0)]\omega_2 \log q(1)],$$

since all the stock in the final period (t = 1) will be consumed. The partial derivatives of v with respect to the various unknowns are

$$(3.8)\qquad \frac{\partial v}{\partial c_i(0)} = \frac{\omega_i}{c_i(0)}, \quad \frac{\partial v}{\partial x_i(0)} = \frac{\delta\omega_1\alpha_i}{x_i(0)}, \qquad i = 1,\ 2.$$

Because of the constraints (3.5) on available stock, one must have

$$(3.9) \qquad \frac{\partial v}{\partial c_i(0)} = \frac{\partial v}{\partial x_i(0)}, \qquad i = 1, 2,$$

or, from (3.8),

$$(3.10) \qquad \frac{\omega_i}{c_i(0)} = \frac{\delta\omega_1\alpha_i}{x_i(0)}, \qquad i = 1, 2.$$

The solution of (3.10) is

$$(3.11) \qquad \begin{aligned} c_1(0) &= \left(\frac{1}{1+\delta\alpha_1}\right)z_1(0), \quad x_1(0) = \left(\frac{\delta\alpha_1}{1+\delta\alpha_1}\right)z_1(0), \\ c_2(0) &= \left(\frac{\omega_2}{\omega_2+\delta\omega_1\alpha_2}\right)z_2(0), \quad x_2(0) = \left(\frac{\delta\omega_1\alpha_2}{\omega_2+\delta\omega_1\alpha_2}\right)z_2(0). \end{aligned}$$

Substitution of (3.11) in expression (3.7) for welfare gives

$$(3.12) \qquad \begin{aligned} G_1[z(0),q(1)] = {} & (1 + \delta\alpha_1)\omega_1 \log z_1(0) \\ & + (\delta\alpha_2\omega_1 + \omega_2) \log z_2(0) + K_1[q(1)], \end{aligned}$$

where K_1 does not depend upon $z(0)$. Equation (3.12) can be re-written:

$$(3.13) \qquad \begin{aligned} G_1[z(0),q(1)] = {} & \omega_1(1) \log z_1(0) + \omega_2(1) \log z_2(0) \\ & + K_1[q(1)], \end{aligned}$$

where

$$(3.14) \qquad \begin{aligned} \omega_1(1) &= (1 + \delta\alpha_1)\omega_1, \\ \omega_2(1) &= \delta\alpha_2\omega_1 + \omega_2. \end{aligned}$$

Note that G_1 is similar to G_0, except that the "weights" ω_i are replaced by new weights $\omega_i(1)$, and a term independent of $z(0)$ has been added. Note, too, that the new weights are linear functions of the old ones. These are the key features that will enable us to go easily from a problem with horizon (T-1) to a problem with horizon T.

In general, one has the following relation between G_T and G_{T-1}:

$$(3.15)\qquad G_T[z(0),q(1),\ldots,q(T)] = \max\Big\{\omega_1 \log c_1(0) + \omega_2 \log c_2(0) + \delta G_{T-1}[z(1),q(2),\ldots,q(T)]\Big\} .$$

where the maximum is subject to (3.5).

Using an argument by induction similar to that used to obtain G_1, one can show without much trouble that

$$(3.16)\qquad G_T[z(0),q(1),\ldots,q(T)] = \omega_1(T) \log z_1(0) + \omega_2(T) \log z_2(0) + K_T[q(1),\ldots,q(T)],$$

where

$$\omega_1(T) = \omega_1 + \delta\alpha_1\omega_1(T-1),$$

$$(3.17)\qquad \omega_2(T) = \omega_2 + \delta\alpha_2\omega_1(T-1),$$

$$\omega_i(0) = \omega_i, \qquad\qquad i = 1,\ 2.$$

Equations (3.17) can be solved to give

$$\omega_1(T) = \sum_0^{T-1} (\delta\alpha_1)^t\omega_1,$$

$$(3.18)$$

$$\omega_2(T) = \omega_2 + \delta\alpha_2 \sum_0^{T-2} (\delta\alpha_1)^t\omega_1.$$

As in the case of Section 2, the formula for K_T is irrelevant for the determination of the optimal policy, but for completeness I give it anyway.

$$(3.19)\quad K_T[q(1),\ldots,q(T)] = \omega_1 \log \omega_1 + \omega_2 \log \omega_2 + \delta\alpha_1\omega_1(T-1) \log \delta\alpha_1\omega_1(T-1)$$
$$+ \delta\alpha_2\omega_1(T-1)\log \delta\alpha_2\omega_1(T-1) - \omega_1(T)\log \omega_1(T)-\omega_2(T)\log\omega_2(T)$$
$$+ \delta[\omega_1(T-1)\beta\omega_2(T-1)q(1)] + \delta K_{T-1}[q(2),\ldots,q(T)],$$
$$K_0 = 0.$$

Consumption and Saving

In deriving (3.16)-(3.18), the optimal consumptions and inputs in period 0 will be found to be:

$$(3.20)\quad \begin{aligned} c_1(0) &= \left[\frac{\omega_1}{\omega_1(T)}\right] z_1(0), \quad x_1(0) = \left[\frac{\omega_1(T)-\omega_1}{\omega_1(T)}\right] z_1(0), \\ c_2(0) &= \left[\frac{\omega_2}{\omega_2(T)}\right] z_2(0), \quad x_2(0) = \left[\frac{\omega_2(T)-\omega_2}{\omega_2(T)}\right] z_2(0). \end{aligned}$$

Hence, the optimal consumptions and inputs in period t of a program with horizon T will be

$$(3.21)\quad \begin{aligned} c_1(t) &= \left[\frac{\omega_1}{\omega_1(T-t)}\right]z_1(t), \quad x_1(t) = \left[\frac{\omega_2(T-t)-\omega_2}{\omega_1(T-t)}\right]z_1(t), \\ c_2(t) &= \left[\frac{\omega_2}{\omega_2(T-t)}\right]z_2(t), \quad x_2(t) = \left[\frac{\omega_2(T-t)-\omega_2}{\omega_2(T-t)}\right]z_2(t). \end{aligned}$$

Infinite Programs

If $\delta\alpha_1 < 1$, then the sums in (3.18) converge as $T \to \infty$, and $\omega_1(T)$ converges to, say, $\tilde{\omega}_1$, given by

$$(3.22)\quad \begin{aligned} \tilde{\omega}_1 &= \left(\frac{1}{1-\delta\alpha_1}\right)\omega_1 , \\ \tilde{\omega}_2 &= \left(\frac{\delta\alpha_2}{1-\delta\alpha_1}\right)\omega_1 + \omega_2 . \end{aligned}$$

Hence consumptions and inputs in <u>every</u> period will be

$$
(3.23)\quad
\begin{aligned}
c_1(t) &= (1-\delta\alpha_1)z_1(t) && \equiv \gamma_1 z_1(t)\\
x_1(t) &= \delta\alpha_1 z_1(t) && \equiv \sigma_1 z_1(t)\\
c_2(t) &= \left[\frac{\omega_2(1-\delta\alpha_2)}{\omega_1\delta\alpha_2 + \omega_2(1-\delta\alpha_1)}\right]z_2(t) && \equiv \gamma_2 z_2(t)\\
x_2(t) &= \left[\frac{\omega_1\delta\alpha_2}{\omega_1\delta\alpha_2 + \omega_2(1-\delta\alpha_1)}\right]z_2(t) && \equiv \sigma_2 z_2(t).
\end{aligned}
$$

Notice that the consumption and saving coefficients do not depend on the sequence $q(t)$ of primary resources. Notice also that the consumption and saving coefficients γ_1 and σ_1 for commodity 1 do not depend upon the weights ω_1 and ω_2 in the welfare function. The saving coefficient σ_2 for commodity 2 is an increasing function of the social time discount factor.

For $\delta = 1$, we have

$$
(3.24)\quad
\begin{aligned}
\sigma_1 &= \alpha_1,\\
\sigma_2 &= \omega_1.
\end{aligned}
$$

For δ close to 1 we have the approximation

$$
(3.25)\quad \sigma_2 \approx \omega_1\left[1-(1-\delta)\omega_2(1 + \frac{\alpha_1}{\alpha_2})\right].
$$

Consumption of both commodities approaches zero as $\delta \to (1/\alpha_1)$.

I turn now to a discussion of the time pattern of output of commodity 1 for optimal infinite programs. Let

$$
(3.26)\quad \xi_1 = \log \sigma_1 ,
$$

so that

$$
(3.27)\quad \log x_1(t) = \xi_1 + \log z_1(t)
$$

(note that $\xi_1 < 0$); and let

$$(3.28) \qquad \zeta = \beta + \alpha_1 \xi_1 + \alpha_2 \xi_2 .$$

I will show that, for an optimal infinite program,

$$(3.29) \qquad \log z_1(t) = \alpha_1^t \log z(0) + \left(\sum_0^{t-1} \alpha_1^k\right)\zeta + \alpha_2 \sum_{k=0}^{t-1} \alpha_1^k \log q(t-1-k).$$

(We have also, of course, $z_2(t) = q(t)$.)

The first term on the right side of (3.29) converges to 0. The second converges to

$$\frac{\zeta}{1-\alpha_1} = \frac{\zeta}{\alpha_2} .$$

The third term depends upon the behavior of $q(t)$.

Consider the special case in which $q(t)$ grows (or declines) at a constant rate:

$$(3.30) \qquad q(t) = q(0)q^t.$$

This includes the special case of constant $q(t)$. It can be shown that the third term now will be asymptotically $t \log q$, so that

$$(3.31) \qquad z_1(t) \sim hq^t,$$

where h is some constant depending on the various parameters.+ In particular, if $q(t)$ is constant ($q = 1$), then

$$(3.32) \qquad \lim_{t\to\infty} z_1(t) = q(0) \exp\left(\frac{\zeta}{\alpha_2}\right).$$

+For two functions $f(t)$ and $g(t)$ of t, $f(t) \sim g(t)$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

Hence, for exponential growth of $q(t)$, the optimal program approaches asymptotically a proportional growth path in which output of commodity 1, consumption of both commodities, and inputs of both commodities are all growing at the same rate as $q(t)$. In particular, consumption will be asymptotically

$$(3.33) \qquad c_i(t) \sim k_i q^t,$$

for some constants k_i. The one period welfare in period t will then be approximately

$$(3.34) \qquad v(t) \approx \bar{v}(t),$$

where

$$(3.35) \qquad \bar{v}(t) \equiv \omega_1 \log k_1 + \omega_2 \log k_2 + t \log q.$$

The precise meaning of (3.34) is that

$$(3.36) \qquad \lim_{t\to\infty} [v(t) - \bar{v}(t)] = 0,$$

even though $v(t)$ and $\bar{v}(t)$ may be increasing without limit.

One can show that the constant

$$(3.37) \qquad \omega_1 \log k_1 + \omega_2 \log k_2$$

is maximum when $\delta = 1$ (see Exercise 1 at the end of this section).

To complete this section I will derive formula (3.29) for the output $z_1(t)$ of commodity 1. From (3.27), (3.1) and (3.5) we have

$$(3.38) \qquad \log z_1(t+1) = \beta + \alpha_1[\xi_1 + \log z_1(t)] + \alpha_2[\xi_2 + \log z_2(t)].$$

Using (3.28) and the fact that $z_2(t) = q(t)$, we can rewrite (3.38) as

$$(3.39) \qquad \log z_1(t+1) = \zeta + \alpha_1 \log z_1(t) + \alpha_2 \log q(t).$$

It is not hard to verify that (3.29) is the solution of the difference equation (3.39).

Exercise 1. In the case of constant $q(t) = q$, write the equations for a stationary state for given consumption and savings coefficients γ_1 and σ_1. Find the values of these coefficients that maximize the (constant) one-period welfare

$$\omega_1 \log c_1 + \omega_2 \log c_2 ,$$

and verify that the resulting stationary state is the limit as $t \to \infty$ of the optimal program for $\delta = 1$.

Exercise 2. Let $g = q-1$ be the (constant) rate of growth of $q(t)$, and let r be the asymptotic shadow interest rate. Show that

$$\frac{1+r}{1+g} = \frac{\dfrac{\omega_1}{\gamma_1}}{\dfrac{\omega_1\sigma_1}{\gamma_2} + \dfrac{\omega_2\sigma_2}{\gamma_2}} = 1 + \frac{\omega_1 - \sigma_2}{\gamma_2\left[\omega_1\left(\dfrac{\sigma_1}{\gamma_1}\right) + \omega_2\left(\dfrac{\sigma_2}{\gamma_2}\right)\right]} ,$$

and hence that $r > g$ for $\delta < 1$, and $r = g$ for $\delta = 1$.

4. Multicommodity Case: No Primary Resources

In this section I discuss the case in which there are two groups of commodities:

(a) new, which are produced according to linear-logarithmic production functions, and

(b) second-hand, which are classified according to age as well as other physical characteristics, and which are used up according to given distributions of lifetime (which may have any form).

This model of production was discussed fully in Chapter I, Section 7. Since the introduction of primary resources into the system leads to results that are qualitatively different in some respects, I defer that case to the next section.

The present case may be regarded as a multicommodity generalization of the one-commodity case discussed in Section 2. One must now determine in each period, not only the optimal allocation of the stock of each commodity between consumption and input into production, but one must also allocate the total input of a commodity among the several production sectors (each of which produces a single "new" commodity).

The results for this case were described qualitatively in Section 1 of this chapter. Here I will give a more detailed and mathematical description of the results, but without proof. The proof combines the dynamic programming technique of the previous two sections with the type of calculation used in Chapter III, Section 1.2.

For the convenience of the reader, let me first restate the production model as compactly as possible (see Chapter I, Section 7, equations (7.6)-(7.9)).

Let there be M different commodities altogether, the first N of them being the newly produced ones. Let x_i and y_i, respectively, be the input and output of commodity i ($i = 1, \ldots, M$), and define

$$X_i = \log x_i \,, \qquad Y_i = \log y_i \,, \tag{4.1}$$

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_M \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_M \end{pmatrix}.$$

The "production function", for all commodities, is

$$Y = \beta + \eta(f) + A'X \tag{4.2}$$

where β is a vector of parameters, A is an MxM matrix $((\alpha_{ij}))$ of parameters, f is an MxN matrix $((f_{ij}))$, and η is a certain vector-valued function of f.

For $1 \leqq i \leqq M$ and $1 \leqq j \leqq N$, α_{ij} is the non-negative elasticity of production of commodity j with respect to good i. For $N + 1 \leqq j \leqq M$, α_{ij} is 1 if j is the "same" commodity as i, but one period older; otherwise $\alpha_{ij} = 0$. We have $\alpha_{ij} \geqq 0$ for all i, j. I assume that the production of new goods exhibits <u>non-increasing returns to scale</u>. Mathematically, the important implications of these various remarks about the α_{ij} can be summarized by

$$\alpha_{ij} = 0, \sum_{k=1}^{M} \alpha_{kj} \leqq 1, \qquad \text{all } i \text{ and } j. \tag{4.3}$$

For $1 \leqq i \leqq M$ and $1 \leqq j \leqq N$, f_{ij} is the proportion of the total input x_i of commodity i that is devoted to the production of commodity j. Thus the f_{ij} are production allocation variables to be determined in the program. By definition,

$$f_{ij} \geqq 0, \sum_{k=1}^{N} f_{ij} = 1. \tag{4.4}$$

The function η is defined by+

$$\eta_j(f) = \begin{cases} \sum_{i=1}^{M} \alpha_{ij} \log f_{ij}, & 1 \leqq j \leqq N, \\ 0, & N + 1 \leqq j \leqq M, \end{cases}$$

$$\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix} . \tag{4.5}$$

For a social welfare function, I take the linear-logarithmic form, with time discounting, of Chapter II, Section 3. Let $c(t)$ be as usual the consumption vector for period t, and define

$$C_i(t) = \log c_i(t), \tag{4.7}$$

$$C(t) = \begin{pmatrix} C_1(t) \\ \vdots \\ C_M(t) \end{pmatrix} . \tag{4.8}$$

+The convention that $0 \log 0 = 0$ is to be understood here.

Let ω be a vector of M non-negative components ω_i, and let δ be the social time discount factor; then the social welfare function is assumed to be of the form

$$(4.9) \qquad \sum_{t=0}^{T} \delta^t \omega' C(t), \qquad \text{with } \delta < 1.$$

An <u>optimal</u> program is one that maximizes the welfare (4.9), given the initial stock vector z(0), subject to+

$$(4.10) \qquad \begin{aligned} &Z(t+1) = \beta + \eta[f(t)] + A'X(t), && 0 \leqq t \leqq T-1 \\ &c(t) + x(t) \leqq z(t), && \\ &c(t) \geqq 0, \quad x(t) \geqq 0, && 0 \leqq t \leqq T, \end{aligned}$$

and to condition (4.4) on the $f_{ij}(t)$.

To describe the optimal program, let $G_T[z(0)]$ denote the maximum welfare possible for a program with horizon T, given the initial stock z(0). Then it can be shown that

$$(4.11) \qquad G_T[z(0)] = \omega(T)'Z(0) + K_T,$$

where

$$(4.12) \qquad \omega(t) = \sum_{k=0}^{t} (\delta A)^k \omega,$$

and K_T is a quantity that depends upon T but not upon z(0). In particular, $K_0 = 0$. Note that $G_0[z(0)] = \omega' Z(0)$, which is obvious since for T = 0 all the initial stock will be consumed.

It can also be shown that in an optimal program with horizon T, consumption, inputs, and input allocation are determined by

$$(4.13) \qquad x_i(t) = \left[\frac{\omega_i(T-t) - \omega_i}{\omega_i(T-t)}\right] z_i(t),$$

$$(4.14) \qquad c_i(t) = \left[\frac{\omega_i}{\omega_i(T-t)}\right] z_i(t),$$

+Again, $Z_i(t) = \log z_i(t)$.

$$(4.15)\qquad f_{ij}(t) = \frac{\alpha_{ij}\omega_j(T-1-t)}{\sum_{k=1}^{N} \alpha_{ik}\omega_k(T-1-t)} .$$

Notice that the optimal production proportions in (4.13)-(4.15) do not depend upon current stocks.

Programs with Infinite Horizons. If the horizon T is increased without bound, equations (4.11) and (4.12) approach limits, which characterize the optimal program for an infinite horizon. Define

$$(4.16)\qquad \begin{aligned} \tilde{A} &= \sum_{k=0}^{\infty} (\delta A)^k, \\ \tilde{\omega} &= \tilde{A}\omega. \end{aligned}$$

The optimal program for an infinite horizon is determined by

$$(4.17)\qquad x_i(t) = \left[\frac{\tilde{\omega}_i - \omega_i}{\tilde{\omega}_i}\right] z_i(t) \equiv \sigma_i z_i(t),$$

$$(4.18)\qquad c_i(t) = \left[\frac{\omega_i}{\tilde{\omega}_i}\right] z_i(t) \equiv \gamma_i z_i(t),$$

$$(4.19)\qquad f_{ij}(t) = \frac{\alpha_{ij}\tilde{\omega}_j}{\sum_{k=1}^{N} \alpha_{ik}\tilde{\omega}_k} \equiv \phi_{ij}$$

Notice that the optimal proportions σ_i, γ_i, and ϕ_{ij} are constant. Define

$$(4.20)\qquad \xi_i = \log \sigma_i, \quad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_M \end{pmatrix} ,$$

$$(4.21)\qquad \zeta = \beta + \eta(\phi) + A'\xi.$$

It follows from (4.10) that, for the optimal program, $Z(t)$ is given in terms of $Z(t-1)$ by

$$(4.22) \qquad Z(t) = \zeta + A'Z(t-1), \qquad t \geq 1.$$

The solution of this difference equation is

$$(4.23) \qquad Z(t) = \sum_{k=0}^{t-1} (A')^k \zeta + (A')^t Z(0).$$

The sequence $Z(t)$ may converge or diverge as t gets large; i.e., the sequence of stocks $z_i(t)$ may converge to a non-zero limit, diverge, or converge to zero. However, if the matrix A of coefficients α_{ij} satisfies certain further conditions, then the relative proportions of the quantities $z_i(t)$ do tend towards limits as t increases; indeed, $z(t)$ approaches a <u>proportional growth path with a constant rate of growth</u>.

The mathematical conditions that I have in mind have the following economic interpretation:

(a) There are <u>constant</u> returns to scale in the production of each commodity $j = 1, \ldots, N$.

(b) The economy cannot be decomposed into independent sub-economies.

(c) Production is acyclic in the sense that one <u>cannot</u> partition the commodities into groups $B_1, \ldots, B_K$ such that commodities in group B_2 can be produced from commodities in group B_1 only, commodities in B_3 can be produced from commodities in B_2 only, ..., etc., and commodities in B_1 can be produced from commodities in B_K only.

Mathematically, conditions (a)-(c) are expressed by

$$(a') \quad \sum_{i=1}^{M} \alpha_{ij} = 1, \qquad \text{all } j$$

(this corresponds to (a) above).

(b') A is fully regular+ (this corresponds to (b) and (c)).

+See GANTMACHER, Vol. 2, p. 88.

As a consequence of conditions (a') and (b'), the limit

$$(4.24) \qquad \bar{A} = \lim_{k\to\infty} A^k$$

exists; furthermore, all of the columns of $\bar{A}$ are identical, say equal to the (non-negative) vector $\bar{a}$, and $\sum_i \bar{a}_i = 1$. The asymptotic growth factor of the optimal program is

$$(4.25) \qquad (1 + g) = e^{\bar{a}'\zeta},$$

g being the asymptotic growth rate. The growth factor (1 + g) can also be expressed as a product of two factors:

$$(4.26) \qquad (1 + g) = \left[\prod_{i=1}^{M} \sigma_i^{\bar{a}_i}\right][e^{\bar{a}'\beta+\bar{a}'\eta(\phi)}].$$

The first factor is a geometric mean of the "saving coefficients" σ_i of the several commodities (see (4.17)). The second factor is the growth factor that would be realized if consumption were reduced to zero, but the same allocation coefficients ϕ_{ij} were used (see (4.18)). Equation (4.26) corresponds to equation (2.18) of Section 2 for the one-commodity case, with the first factor of (4.26) corresponding to the saving coefficient δ in (2.18), and the second factor in (4.26) corresponding to e^{β}.

As the social time discount factor δ approaches 1, the consumption coefficients γ_i in (4.18) approach 0, and the saving coefficients σ_i approach 1. Furthermore, the allocation coefficients ϕ_{ij} approach the values

$$(4.27) \qquad \lim_{\delta\to 1} \phi_{ij} = \frac{\alpha_{ij}\,\bar{a}_j}{\bar{a}_i} \equiv \bar{\phi}_{ij} \; .$$

Notice that <u>in this limiting case the consumption, saving, and allocation coefficients depend only upon technological parameters, and not upon the weights</u> ω_i <u>in the social welfare function</u>.

Further light on this point is obtained by looking at what happens to the weights $\tilde{\omega}_i$ as δ approaches 1. Looking at (4.16), we see that the sum defining $\breve{A}$ diverges for $\delta = 1$, so that $\tilde{\omega}$ is not defined in this case. However, it can be shown that

$$(4.28) \qquad \lim_{\delta \to 1} (1-\delta) \sum_{0}^{\infty} (\delta A)^k = \bar{A},$$

and also+

$$(4.29) \qquad \bar{A}\omega = \bar{a},$$

so that, for δ close to 1, $\tilde{\omega}$ is approximately proportional to $\bar{a}$. Hence for δ close to 1, the "dynamic programming valuation function" G_{∞} (see (4.11)) evaluates current stocks with weights $\tilde{\omega}_i$ that are approximately proportional to the $\bar{a}_i$, i.e. with weights that are approximately independent of the original weights ω_i in the social welfare function.

It can also be shown that as δ approaches 1 the asymptotic rate of growth in (4.25) approaches the maximum possible rate of growth of output. This case thus provides an example of the so-called "turnpike theorem" (see Chapter V). It should be emphasized that this maximal growth factor is typically larger than the second factor in (4.26).

Now I consider the long-run direction of the optimal path, i.e., the long-run relative proportions of the commodities. Define

$$(4.30) \qquad \zeta^* = \sum_{0}^{\infty} (A'-\bar{A}')^k \zeta , \qquad \Pi_j = e^{\xi_j^*} ;$$

then, in the long run, the stocks $z_j(t)$ are in the same relative proportions as the quantities Π_j. To be precise, let g be the asymptotic growth rate of the optimal program, as given by (4.25), and define

$$(4.31) \qquad \log h_0 = \bar{a}'Z(0).$$

+Note that $\bar{A}v = \bar{a}$ for any vector v such that $\sum_{i} v_i = 1$.

It can be shown that

$$(4.32) \qquad z_j(t) \sim h_0(1+g)^t \Pi_j, \qquad j = 1, \ldots, M.$$

Note that the long-run proportions Π_j depend upon the various parameters of the problem. In particular, as δ approaches 1, the Π_j approach the proportions along the path of proportional growth with the highest rate of growth (again, see the discussion of the "turnpike theorem" in Chapter V).

Finally, one can show that the asymptotic shadow interest rate is greater than the asymptotic growth rate if $\delta < 1$, and equals the growth rate if $\delta = 1$.

5. Multicommodity Case with Primary Resources

In this section I expand the model of the previous section to include a third group of commodities, primary resources. These are commodities that are not produced, but whose stocks are determined exogenously in each period, this sequence of stocks being independent of the program chosen. Whether or not a particular commodity should be classified as a primary resource will typically depend upon the circumstances of the problem. For example, in a very poor country the population growth (or decline) may depend upon which economic program is chosen, whereas in a rich country the population might well be taken to be a primary resource, at least as a good approximation. Land should typically be treated as a primary resource, unless the economic programs considered involve possible long-run changes in the fertility of the soil, etc.

The formulas describing the optimal programs for this case are similar to those for the case of no primary resources. However, the evolution of the output of the produced (i.e. non-primary) resources will depend upon the availability of primary resources. For example, if the supply of primary resources is constant, then in an optimal program all outputs, consumption, etc. will approach constant levels in the long run. On the other hand, if the various primary resources are growing exponentially, then output and consumption in the various sectors will also be asymptotically exponential, with possibly different rates for different sectors.

Suppose now that to the list of M newly produced and second-hand commodities, we add P commodities, called primary resources, which enter into the production of the new commodities, but which are themselves exogenously supplied. Let $z_{(1)}(t)$ denote the vector of stocks of the produced commodities (1 to M), and $z_{(2)}(t)$ denote the vector of stocks of primary resources (M+1 to M+P), at the beginning of period t. Let $Z_{(1)}(t)$ and $Z_{(2)}(t)$ denote the corresponding vectors of logarithms. Using the notation of Section 4, I assume

$$(5.1) \qquad Z_{(1)}(t) = \beta + \eta(f[t]) + A'X(t).$$

Note, however, that here the matrix A has (M+P) rows and M columns and the matrix $f[t] = ((f_{ij}[t]))$ of allocation coefficients has (M+P) rows and N columns. Conditions (4.3) and (4.4) are still satisfied (with the appropriate minor modifications due to the change in the number of commodities).

By definition, the stock of primary resources is determined by

$$(5.2) \qquad Z_{(2)}(t) = Q(t),$$

where Q(t) is a given sequence.

It can be shown that the maximum welfare obtainable, given the initial stocks z(0) and the primary resource sequence q(1), q(2), etc., is

$$(5.3) \qquad \begin{aligned} G_T[z(0),q(1),\ldots,q(T)] = {} & \omega_{(1)}(T)'Z_{(1)}(0) \\ & + \omega_{(2)}(T)'Z_{(2)}(0) \\ & + K_T[q(1),\ldots,q(T)], \end{aligned}$$

where K_T is independent of $z(0)$ (I omit the lengthy formula for K_T), and $\omega_{(1)}(T)$ and $\omega_{(2)}(T)$ are given by

$$(5.4)\qquad \begin{aligned} \omega_{(1)}(T) &= \omega_{(1)} + \delta A_1\omega_{(1)}(T-1), \\ \omega_{(2)}(T) &= \omega_{(2)} + \delta A_2\omega_{(1)}(T-1), \\ \omega_{(1)}(0) &= \omega_{(1)}, \end{aligned}$$

and

$$\omega_{(1)} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_M \end{pmatrix}, \qquad \omega_{(2)} = \begin{pmatrix} \omega_{M+1} \\ \vdots \\ \omega_{M+P} \end{pmatrix},$$

$$(5.5)\qquad \begin{cases} A_1 = ((\alpha_{ij}))_{\substack{i=1,\dots,M \\ j=1,\dots,M}} \\ \\ A_2 = ((\alpha_{ij}))_{\substack{i=M+1,\dots,M+P \\ j=1,\dots,M.}} \end{cases}$$

Equations (5.4) can be solved to give

$$(5.6)\qquad \begin{cases} \omega_{(1)}(T) = \sum_{k=0}^{T} (\delta A_1)^k \omega_{(1)}, \\ \\ \omega_{(2)}(T) = \omega_{(2)} + \delta A_2 \sum_{k=0}^{T-1} (\delta A_1)^k \omega_{(1)}. \end{cases}$$

If we define the vector $\omega(t)$ by

$$(5.7)\qquad \omega(t) = \begin{bmatrix} \omega_{(1)}(t) \\ \\ \omega_{(2)}(t) \end{bmatrix},$$

then the optimal program is given by equations (4.13)-(4.15) of Section 4, using, however, the formulas (5.6) and (5.7) for the vector $\omega(t)$ (instead of the formula (4.12) of Section 4).

<u>Programs with Infinite Horizons</u>. If the horizon T increases without bound, the sums in (5.6) will converge for all values of δ less than some value $\bar{\delta}$. Typically, if there are non-increasing returns to scale, then $\bar{\delta}$ will be <u>greater</u> than unity. Of course, if $A_2 = 0$, then primary resources do not enter into the production of the produced commodities, and one is essentially back in the special case of Section 4.

Letting ω denote the limit of $\omega(T)$ as T gets large, one has from (5.6)

$$(5.8)\qquad \begin{aligned} \tilde{\omega}_{(1)} &= \sum_{k=0}^{\infty} (\delta A_1)^k \omega_{(1)}, \\ \tilde{\omega}_{(2)} &= \omega_{(2)} + \delta A_2 \sum_{k=0}^{\infty} (\delta A_1)^k \omega_{(1)}. \end{aligned}$$

With $\tilde{\omega}$ as given by (5.8), one can now use equations (4.17)-(4.19) to describe the optimal program for the case of an infinite horizon. Note that again the <u>proportion</u> of stocks that go to production and consumption are constant in time (but typically different for different commodities).

Defining ζ as in (4.20) and (4.21), the evolution of the beginning-of-period stocks of commodities 1 through M is determined by

$$(5.9)\qquad Z_{(1)}(t) = \zeta + A_1' Z_{(1)}(t-1) + A_2' Q(t-1).$$

The solution of this difference equation is

$$(5.10)\qquad Z_{(1)}(t) = \sum_{0}^{t-1} (A_1')^k \zeta + (A_1')^t Z_{(1)}(0) + \sum_{0}^{t-1} (A_2 A_1^k)' Q(t-1-k).$$

These last two equations correspond to equations (4.22) and (4.23) of Section 4; they are also the generalizations to the multicommodity case of equations (3.38) and (3.29) of Section 3.

Assume again that constant returns to scale prevail,+ and

+The results that follow typically hold even if there are some, but not too strong, increasing returns to scale. What is required is that the largest root of A_1 be less than unity.

that A_1 is fully regular (see the discussion preceding equation (4.24) in Section 4). Then the first term on the right side of (5.10) will converge to a constant vector, and the second term will converge to zero. The behavior of the third term depends upon the behavior of the sequence $Q(t)$.

Suppose that the supply of each primary resource grows at a constant rate; i.e., suppose that

$$(5.11) \qquad q_i(t) = q_i(0)q_i^t$$

or

$$(5.12) \qquad Q(t) = Q(0) + tQ.$$

Then it can be shown that the third term of (5.10) equals

$$(5.13) \qquad \left[\sum_{0}^{t-1} A_2A_1^k\right]' q(0) - [A_2(I-A_1)^{-2}(I-A_1^t)A_1]'Q + t[A_2(I-A_1)^{-1}]'Q.$$

As t gets large, the first two terms of (5.13) approach constant vectors. The third term is of course proportional to t. Define

$$(5.14) \qquad S = [A_2(I-A_1)^{-1}]'Q.$$

One may summarize the situation by saying that

$$(5.15) \qquad \lim_{t\to\infty} (Z_{(1)}(t) - tS) = H,$$

where H is some constant vector. Define

$$(5.16) \qquad h_i = e^{H_i}, \quad s_i = e^{S_i}, \qquad i = 1, \ldots, M;$$

then (5.15) can be rewritten

$$(5.17) \qquad z_i(t) \sim h_i s_i^t \,.$$

In other words, <u>in the long run the beginning-of-period stock of each commodity</u> — and therefore consumption, too — <u>will tend to grow exponentially</u>. The asymptotic growth rates for different commodities will typically be different. However, if all of the primary resources grow at the <u>same</u> rate, then in the long run all of the produced commodities will grow at the same rate, too. [This last point follows from the fact that $A_2(I-A_1)^{-1}$ is a non-negative matrix with all of its column sums equal to unity.]

REFERENCES FOR CHAPTER IV

F. R. GANTMACHER, <u>The Theory of Matrices</u>, 2 vols., New York: Chelsea, 1960.

CHAPTER V

PROPORTIONAL GROWTH PROGRAMS

1. Introduction

A proportional growth program is a feasible program in which all beginning-of-period stocks, consumption, and inputs grow exponentially at the same rate. In such a program, the relative proportions of the stocks, consumption, and inputs remain constant, even though the absolute magnitudes are increasing. Proportional growth (sometimes called "balanced growth") is a natural generalization of the stationary state. If we are interested in growing as "fast" as possible, while maintaining some desirable proportions among the various commodities, then it may appear useful to concentrate on the study of proportional growth, even though in principle the more logical approach would be to search for optimal programs using the "desired proportions" welfare function of Chapter II, Section 3. Finally, theoretical research to date indicates that there may be many circumstances in which efficient or optimal programs tend towards proportional growth in the long run.

Formally, a program $\{z(t), c(t), x(t)\}$ is a proportional growth program (PGP) if, for some non-negative vectors z, c, and x, and some number $g > -1$,

$$\begin{aligned} z(t) &= (1+g)^t z \\ c(t) &= (1+g)^t c \qquad\qquad t = 0, 1, 2, \text{etc.} \\ x(t) &= (1+g)^t x \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} & c + x = z \\ & [(1+g)^t x, (1+g)^{t+1} z] \text{ is in } \mathcal{T}, \; t = 0, 1, \text{etc.}, \end{aligned} \tag{1.2}$$

where $\mathcal{T}$ is the production possibility set.

Equations (1.1) express the proportional growth of the program, and conditions (1.2) its feasibility (see Chapter I, Section 3). The number g is called the <u>growth rate</u> of the program. I will denote a PGP determined as in (1.1)-(1.2) by the quadruple (z, c, x, g).

To be precise, a program described by (1.1) and (1.2) should be called a constant rate PGP, since the relative proportions of the commodities would also remain the same if the factor $(1 + g)^t$ were replaced by any other function of time. However, in these notes I will consider only constant rate PGP so that the qualifying phrase "constant rate" will not be used.+

In this chapter I summarize three theoretical propositions about PGP's. The first two of these concern the rate of interest for "best" PGP's. The first states that for the fastest growing PGP without (non-technological) consumption, the interest rate equals the growth rate. The second states that for an efficient PGP with consumption, the interest rate exceeds the growth rate.

The third result describes a situation in which all consumptionless programs that are "optimal" in a certain sense tend toward the fastest growing PGP (this is the so-called "turnpike theorem").

2. Fastest Growing Proportional Growth Without Consumption

The earliest mathematical study of the relation between the shadow interest rate and the rate of growth in "best" PGP's was that of VON NEUMANN, who studied the case of consumptionless programs; that is, programs in which $c(t) = 0$. It should be emphasized that this does not exclude "technological consumption", e.g. the consumption of food necessary to produce labor (see Chapter II, Section 1).

If we follow the approach used thus far in these notes, we are not prepared to choose among programs with zero consumption. Implicit in von Neumann's treatment of the problem was the idea

---

+Indeed, as far as I am aware, the use of the terms "proportional" or "balanced" growth always has referred to the constant rate case.

that of two consumptionless PGP's, the one with the higher rate of growth of output is the better. Such a point of view might be appropriate in a "crash program" of development in which one reduces consumption to the subsistence level and tries to achieve some given output targets as quickly as possible. Of course, the consumptionless PGP with the highest rate of growth might have outputs in "undesirable" proportions. The proposition that this last problem may not arise for a sufficiently long crash program is the subject of Section 4.

Before stating von Neumann's result, we need some definitions. An input vector x is called <u>balanced</u> if for some number $g > -1$ the input-output pair $(x, [1 + g]x)$ is feasible. The largest number g for which $(x, [1 + g]x)$ is feasible is called the <u>growth rate associated</u> with x (in principle g may be infinite). It is obvious that a PGP must use a balanced input vector.

For input-output pairs that are not proportional, the following concept is a generalization of the growth rate. If (x,y) is an input-output pair, the <u>coefficient of expansion</u> $R(x,y)$ is defined by

$$(2.1) \qquad R(x,y) = \max \left\{ k \mid y \geqq kx \right\} .$$

(The reader has already met the coefficient of expansion under another name in the "desired proportions" welfare function of Chapter II, Section 3.) For a balanced input vector x, with output $(1 + g)x$, the coefficient of expansion is of course equal to $1 + g$, i.e.

$$R(x, [1 + g]x) = 1 + g.$$

A consumptionless PGP is called fastest growing if it has the maximum growth rate possible among all feasible consumptionless PGP's. Von Neumann was concerned not only with demonstrating the existence of fastest growing consumptionless PGP's, but also of characterizing them in terms of shadow prices. A vector p of prices together with a number r is called an <u>equilibrium</u> price-interest pair+ if

$$(2.2) \qquad \begin{aligned} & p \geqq 0, \; r > -1 \\ & p'\left(\frac{y}{1+r} - x\right) \leqq 0 \text{ for all } (x,y) \text{ in } \mathcal{T}. \end{aligned}$$

Von Neumann showed that under certain conditions on the production possibility set $\mathcal{T}$, one has the following proposition: There exist $\hat{x}$, g, and p such that

(a) $\hat{x}$ is balanced, with growth rate g

(b) (p,g) is an equilibrium price-interest pair

(c) $1 + g = \max\limits_{(x,y)\in\mathcal{T}} R(x,y)$.

Thus (a) and (c) assert the existence of a fastest growing PGP, and (b) states that for the corresponding system of shadow prices the interest rate equals the maximum growth rate.

One set of conditions under which von Neumann's proposition is valid is described below; this set is more general than the set originally used by von Neumann himself.

The assumptions are:

(i) <u>Constant returns to scale</u>. If (x,y) is in $\mathcal{T}$, then so is (kx,ky) for any non-negative number k.

(ii) <u>Additivity</u>. If (x,y) and $(\bar{x},\bar{y})$ are in $\mathcal{T}$, then so is $(x + \bar{x},\; y + \bar{y})$.

(iii) <u>Continuity</u>. If every one of a sequence (x_n,y_n) of input-output pairs is in $\mathcal{T}$, and if

$$x = \lim_n x_n, \qquad y = \lim_n y_n,$$

then (x,y) is in $\mathcal{T}$.

+See Section 3 of this chapter for a further discussion of the shadow-price interpretation of an equilibrium price-interest pair.

(iv) Nothing from nothing. There is no feasible pair $(0,y)$ with $y \geqq 0$.

(v) Free disposal. If (x,y) is feasible, and if $\bar{x} \geqq x$ and $\bar{y} \leqq y$, then $(\bar{x},\bar{y})$ is also feasible.

(vi) Every commodity can be produced. For every $i=1,\ldots,M$ there is a feasible pair (x,y) for which the i-th coordinate of y is positive.

For proofs of von Neumann's proposition under assumptions (i)-(vi), see KARLIN, Chapter 9, and GALE.

3. Interest Rates for Efficient Proportional Growth Programs with Consumption

I turn now to the consideration of proportional growth programs with consumption, and in particular to a characterization of efficient PGP's with consumption (see Chapter II, Section 2 for the definition of efficiency).

A shadow interest rate is said to belong to a PGP if, for some non-negative price vector, profit is at a maximum using the given program. To be precise, let r and p denote some interest rate and price vector, respectively, and let (z, c, x, g) be a PGP. The profit is proportional to

$$(3.1) \qquad \frac{(1+g)p'z}{(1+r)} - p'x,$$

since $(1+g)z$ is the output corresponding to the input x. An interest rate $\bar{r}$ is said to belong to the PGP $(\bar{z}, \bar{c}, \bar{x}, \bar{g})$ if there is some non-negative price vector $\bar{p}$ such that

$$(3.2) \qquad \frac{(1+g)\bar{p}'z}{1+\bar{r}} - \bar{p}'x \leqq \frac{(1+\bar{g})\bar{p}'\bar{z}}{1+\bar{r}} - \bar{p}'\bar{x}$$

for all PGP's (z, c, x, g). A PGP may have more than one interest rate belonging to it, or none.

Condition (3.2) can be reformulated as follows, if there are constant returns to scale. The left side of (3.2) must be non-positive, for every program different from the given program, because, if it were positive, then it could be made arbitrarily large by increasing the scale of production. By continuity, the right side must therefore also be non-positive. On the other hand, if the right side of (3.2) were strictly negative, then it could be increased (algebraically) by decreasing the scale of production. Hence the right side of (3.2), i.e. the profit for the given program, must be zero. Condition (3.2) can now be reformulated

$$(3.3) \qquad \begin{aligned} &\frac{(1+\bar{g})\bar{p}'\bar{z}}{1+\bar{r}} - \bar{p}'\bar{x} = 0 \\ &\frac{(1+g)\bar{p}'z}{1+\bar{r}} - \bar{p}'x \leqq 0. \end{aligned}$$

We see, in particular, that $\bar{r}$ is an interest rate belonging to some PGP, if and only if $(\bar{p},\bar{r})$ is an equilibrium price-interest pair, and the PGP to which it belongs gives zero profit [see (2.2)].

Suppose that the production possibility set $\mathcal{J}$ is defined by the Linear Activity Analysis model of Chapter I, Section 5, which, incidentally, satisfies assumptions (i)-(vi) of Section 2 of the present chapter. MALINVAUD has shown that, in this case, if an efficient PGP has an input vector x with all positive coordinates ($x > 0$), then the <u>largest rate of interest belonging to the program is greater than the growth rate of the program</u>.

It is of some interest to compare the present theorem with the von Neumann theorem described in the previous section. In the von Neumann case the rate of growth of the fastest growing PGP is an interest rate belonging to it and typically the only one, i.e., the largest interest rate <u>equals</u> the rate of growth. However, this does not contradict the present result, because in the von Neumann case consumption c is zero although some stocks z_i are positive, so that the von Neumann PGP is not efficient in the sense used in this section.

4. Do Optimal Programs Tend Towards Proportional Growth in the Long Run?

One may say that the program $\{z(t),\ c(t),\ x(t)\}$ tends towards the non-zero proportional growth program (z, c, x, g) in the long run, if

$$\lim_{t \to \infty} \frac{z(t)}{(1+g)^t} = z$$

$$(4.1) \qquad \lim_{t \to \infty} \frac{c(t)}{(1+g)^t} = c$$

$$\lim_{t \to \infty} \frac{x(t)}{(1+g)^t} = x.$$

In this case g is the long-run, or asymptotic, growth rate of the program.

In Chapter IV we saw that the optimal programs for that somewhat special model could tend towards PGP's under certain conditions. The question arises whether this phenomenon generalizes to other models. To my knowledge, theorems of this type for fairly general production possibility sets have been obtained only in the case of consumptionless programs, although the theorems of Chapter IV suggest that the phenomenon is more general. It would appear, however, that the condition of constant returns to scale is crucial here, at least for asymptotic proportional growth at a constant rate.

I will present one theorem along these lines (see RADNER). For other theorems the reader should consult MORISHIMA, McKENZIE, and the papers referred to there. In all of these theorems on consumptionless programs, the PGP to which the optimal programs tend is (or is related to) the fastest growing PGP of the von Neumann theorem. Because of this, such results have been called "turnpike theorems", the "turnpike" being the expansion path of the von Neumann PGP.

I have already noted that the concepts of efficiency and optimality introduced in Chapter II cannot be directly applied to choices among consumptionless programs. However, the same criteria can be defined in terms of final stocks $z(T)$ at the horizon T, instead of in terms of consumption. For example, I will say that a feasible consumptionless program $\bar{z}(0),\ldots,\bar{z}(T)$ <u>is efficient for final stocks</u>, given the initial stock $\bar{z}(0)$, if there is no other feasible program $z(0),\ldots,z(T)$ such that

$$z(0) \leqq \bar{z}(0), \tag{4.2}$$
$$z(T) \geqq \bar{z}(T).$$

Let $\hat{x}$ be the input vector for the fastest growing PGP (which I will suppose here to be unique up to multiplication by positive numbers), and call the set of all non-negative multiples of $\hat{x}$ the <u>von Neumann ray</u>.[+] The fastest growing PGP will expand along the von Neumann ray. The "turnpike theorem" to be presented here states that, roughly speaking, if the horizon T is sufficiently far away, then the path of outputs for any program that is efficient for final stocks will spend most of the time near the von Neumann ray. This situation is depicted in Figure 1, for the case of two commodities.

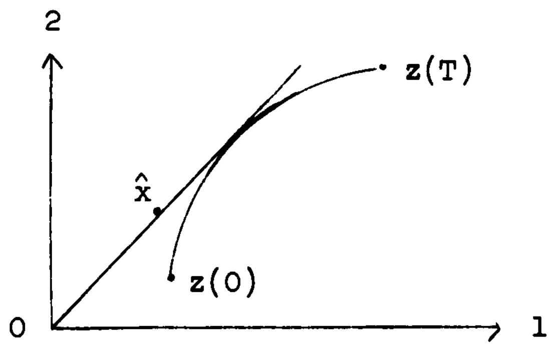


Figure 1

In the figure, the line through the point $\hat{x}$ is the von Neumann ray, and the curved path from $z(0)$ to $z(T)$ represents an efficient program starting from $z(0)$.

[+]This is indeed the ray from the origin passing through $\hat{x}$ in the M-dimensional "commodity space".

In the precise statement of the turnpike theorem that follows, the appropriate concept of "nearness" is one of angular distance rather than ordinary distance. Thus define the following "distance" between two vectors x and $\hat{x}$.

$$(4.3)\qquad d(x,\hat{x}) = \left\|\frac{x}{\|x\|} - \frac{\hat{x}}{\|\hat{x}\|}\right\| ,$$

where, for any vector x, the symbol $\|x\|$ means

$$(4.4)\qquad \|x\| = (\sum_i x_i^2)^{1/2}.$$

The "distance" function (4.3) is interpreted in Figure 2.

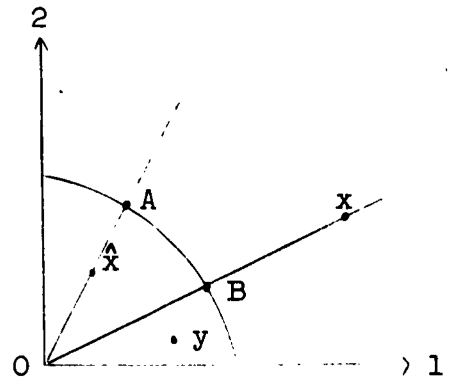


Figure 2

In Figure 2, the point A is the projection of the vector $\hat{x}$ onto the circle of unit radius along the ray from the origin 0 through $\hat{x}$. The point A does in fact represent the vector $\hat{x}/\|\hat{x}\|$. Similarly, the point B represents $x/\|x\|$. The "distance" $d(x,\hat{x})$ is equal to the ordinary distance between the points A and B. Note that the point y in Figure 2 is <u>farther</u> from $\hat{x}$ than x is, in terms of the distance function d.

Roughly speaking, one may say that the distance function d measures the extent to which the <u>relative proportions</u> in two commodity vectors are different.

Suppose now that $\hat{x}$ is, as before, the input vector of the fastest growing PGP, and let (p,g) be the corresponding price-interest pair, as in Section 2. Recall that g is also the growth rate. In addition to the technological assumptions (i)-(vi) of Section 2, suppose that the following conditions are satisfied:+

(vii) $p > 0$

(viii) If an input-output pair (x,y) is not proportional to $(\hat{x}, (1 + g)\hat{x})$, then it yields negative profit, i.e.,

$$\frac{p'y}{1+g} - p'x < 0.$$

(ix) From the initial stock vector $x(0)$ one can get onto the von Neumann ray, i.e., for some number $k > 0$,

$$(z(0), k\hat{x}) \text{ is in } \mathcal{J}.$$

One knows, from considerations similar to those discussed in Chapter III, Section 1, that if a program is efficient for final stocks, given $z(0)$, then for some non-negative vector $\omega \geqq 0$, the program maximizes $\omega' z(T)$ in the set of all feasible programs starting from $z(0)$. Suppose then that such a vector $\omega \geqq 0$ is given, such that $\omega'\hat{x} > 0$.

Turnpike Theorem. Given $z(0)$, for any $\epsilon > 0$ there is a number S such that for any horizon T, and any feasible program $\{z(t)\}$ starting from the given $z(0)$ that maximizes $\omega' z(T)$, the number of periods in which $d(z[t],\hat{x}) \geqq \epsilon$ cannot exceed S.

It should be noted that the number S is independent of the horizon T. A formula for S can also be given (see the original reference).

Finally, I should point out that the turnpike theorem does not hold in general if some assumptions about the technology are not made in addition to (i)-(vi) of Section 2.

+The result in RADNER is proved under somewhat more general conditions than those given here, and also for a somewhat more general concept of optimality.

REFERENCES FOR CHAPTER V

D. GALE, "The closed linear model of production," in Kuhn and Tucker (eds.), Linear Inequalities and Related Systems, Annals of Math. Study No. 38, Princeton N.J.: Princeton University Press, 1956, pp. 285-303.

S. KARLIN, Mathematical Methods and Theory in Games, Programming and Economics, Vol. 1, Reading, Mass.: Addison-Wesley, 1959.

E. MALINVAUD, "On the rate of interest of proportional growth programs," Working Paper No. 32, Management Science Research Center, University of California, Berkeley, August, 1961 (dittoed).

M. MORISHIMA, "Proof of a turnpike theorem, the no joint production case," Review of Economic Studies, Vol. 28 (1961).

L. McKENZIE, "The turnpike theorem for a generalized Leontieff model," Econometrica, 1963 (forthcoming).

R. RADNER, "Paths of economic growth that are optimal with regard only to final states: A turnpike theorem," Review of Economic Studies, Vol. 28 (1961), 98-104.

J. VON NEUMANN, "A model of general economic equilibrium," Review of Economic Studies, Vol. 13 (1945-46), pp. 1-9.

BIBLIOGRAPHY

	Chapter
K. J. ARROW, "The economic implications of learning by doing," Institute for Mathematical Studies in the Social Sciences, Stanford University, ONR Technical Report No. 101, Dec. 7, 1961.	I
K. J. ARROW, H. B. CHENERY, B. S. MINHAS, and R. M. SOLOW, "Capital - labor substitution and economic efficiency," Review of Economics and Statistics, V. 43 (1961), 225-250.	I
K. J. ARROW and L. HURWICZ, "Decentralization and computation in resource allocation," Essays in Economics and Econometrics, Chapel Hill, N.C.: University of North Carolina Press, 1960, 34-104.	III
C. BERGE, Espaces Topologiques, Paris: Dunod, 1959.	III
S. CHAKRAVARTY, "The existence of an optimum savings program," Econometrica, vol. 30 (1962), 178-87.	II
G. B. DANTZIG and P. WOLFE, "The decomposition principle for linear programs," Operations Research, Vol. 8 (1960), 101-111.	III
G. DEBREU, Theory of Value, New York: Wiley, 1959). (Cowles Foundation Monograph 17)	III
G. DEBREU, "Valuation equilibrium and Pareto optimum," Proceedings of the National Academy of Sciences of the U.S.A., Vol. 40 (1954), 588-592.	III
P. A. DOUGLAS, "Are there laws of production?" American Economic Review, Vol. 38 (1948), 1-41.	I
H. G. EGGLESTON, Convexity, Cambridge: The Cambridge University Press, 1958.	III

Chapter

D. GALE, "The closed linear model of production," in Kuhn and Tucker (eds.), Linear Inequalities and Related Systems, Annals of Math. Study No. 38, Princeton N.J.: Princeton University Press, 1956, pp. 285-303. — V

F. R. GANTMACHER, The Theory of Matrices, 2 vols., New York: Chelsea, 1960. — IV

G. HADLEY, Linear Algebra, Reading, Mass.: Addison-Wesley, 1961. — III

L. JOHANSEN, A Multisectoral Model of Economic Growth, Amsterdam: North-Holland, 1960. — I

L. JOHANSEN, "Substitution vs. fixed production coefficients in the theory of economic growth: a synthesis," Econometrica, Vol. 27 (1959), 157-176. — I

S. KARLIN, Mathematical Methods and Theory in Games, Programming and Economics, Vol. 1, Reading, Mass.: Addison-Wesley, 1959. — V

T. C. KOOPMANS (ed.), Activity Analysis of Production and Allocation, New York: Wiley, 1951. — I

T. C. KOOPMANS, Three Essays on the State of Economic Science, Chapter I, New York: McGraw-Hill, 1957. — I,III

W. W. LEONTIEFF, Studies in the Structure of the American Economy, Oxford: Oxford University Press, 1953. — I

E. MALINVAUD, "The analogy between atemporal and intertemporal theories of resource allocation," Review of Economic Studies, Vol. 28 (1961). — III

E. MALINVAUD, "Capital accumulation and efficient allocation of resources," Econometrica, Vol. 21 (1953), 233-268. — I,III

Chapter

E. MALINVAUD, "Efficient capital accumulation: a corrigendum," Working Paper No. 28, Management Science Research Center, University of California, Berkeley, June, 1961. — I,III

E. MALINVAUD, "L'intérêt comme indicateur de choix dans l'allocation des ressources," Association Internationale de Science Economique, Colloque sur la Théorie de l'Intérêt, Royaumont, Mars, 1962 (mimeographed). — III

E. MALINVAUD, "On decentralization in national planning," Working Paper No. 36, Management Science Research Center, University of California, Berkeley, August, 1961. — III,V

E. MALINVAUD, "On the rate of interest of proportional growth programs," Working Paper No. 32, Management Science Research Center, University of California, Berkeley, August, 1961 (dittoed). — V

P. MASSÉ, Le Choix des Investissements, Paris: Dunod, 1959. — III

M. MORISHIMA, "Proof of a turnpike theorem, the no joint production case," Review of Economic Studies, Vol. 28 (1961). — V

L. McKENZIE, "The turnpike theorem, for a generalized Leontieff model," Econometrica, 1963 (forthcoming). — V

A. G. PAPANDREOU, "Resource Allocation for Economic Development," Center of Economic Research, 1962. — II

R. RADNER, "Paths of economic growth that are optimal with regard only to final states: A turnpike theorem," Review of Economic Studies, Vol. 28 (1961), 98-104. — V

	Chapter
R. M. SOLOW, "Investment and technical progress," in Arrow, Karlin, and Suppes (eds.), Mathematical Methods in the Social Sciences, Stanford: Stanford University Press, 1960, pp. 89-104.	I
R. M. SOLOW, "Technical change and the aggregate production function," Review of Economics and Statistics, Vol. 39 (1957), 312-320.	I
J. VON NEUMANN, "A model of general economic equilibrium," Review of Economic Studies, Vol. 13 (1945).	I,V
B. WARD, "Kantorovich on economic calculation," Journal of Political Economy, Vol. 68 (1960), 545-556.	II

BASIC DISTRIBUTION LIST

Nonr-222(77), NR-047-029

University of California

University of Chicago
Statistical Research Center
Chicago, Illinois
Attn: Prof. Paul Meier

Columbia University
Dept. of Industrial Engineering
New York 27, New York
Attn: Prof. Cyrus Derman

Stanford University
Applied Math. and Stat. Lab.
Stanford, California
Attn: Prof. G. J. Lieberman

University of Minnesota
Dept. of Statistics
Minneapolis, Minnesota
Attn: Prof. Ingram Olkin

University of Minnesota
Dept. of Statistics
Minneapolis, Minnesota
Attn: Prof. I. R. Savage

Columbia University
Dept. of Mathematics
New York 27, New York
Attn: Prof. H. Robbins

Florida State University
Dept. of Statistics
Tallahassee, Florida
Attn: Dr. R. A. Bradley

New York University
Institute of Mathematical Sciences
New York 3, New York
Attn: Prof. W. M. Hirsch

Princeton University
Dept. of Mathematics
Princeton, New Jersey
Attn: Prof. S. S. Wilks

Cornell University
Dept. of Plant Breeding
Biometrics Unit
Ithaca, New York
Attn: Walter T. Federer

Columbia University
Dept. of Mathematical Statistics
New York 27, New York
Attn: Prof. T.W. Anderson

University of North Carolina
Statistics Department
Chapel Hill, North Carolina
Attn: Prof. Walter L. Smith

University of California
Dept. of Statistics
Berkeley 4, California
Attn: Prof. J. Neyman

Rutgers - The State University
Statistics Center
New Brunswick, New Jersey
Attn: Prof. Martin B. Wilk

University of Washington
Dept. of Mathematics
Seattle 5, Washington
Attn: Prof. Z. W. Birnbaum

Michigan State University
Dept. of Statistics
East Lansing, Michigan
Attn: Prof. Herman Rubin

Cornell University
Dept. of Mathematics
Ithaca, New York
Attn: Prof. J. Wolfowitz

Brown University
Division of Applied Mathematics
Providence 12, Rhode Island
Attn: Prof. M. Rosenblatt

Harvard University
Dept. of Statistics
Cambridge, Massachusetts
Attn: Prof. W. G. Cochran

University of California
Management Sciences
Research Project
Los Angeles 24, Calif.
Attn: Dr. J. R. Jackson

The Research Triangle Inst.
Stat. Research Division
505 West Chapel Hill St.
Durham, North Carolina
Attn: Dr. M.R. Leadbetter

Columbia University
Dept. of Indust. Engr.
New York 27, New York
Attn: Prof. S.B. Littauer

University of Michigan
Dept. of Mathematics
Ann Arbor, Michigan
Attn: Prof. L. J. Savage

Carnegie Inst. of Technology
Grad. Sch. of Ind. Admn.
Schenley Park
Pittsburgh 13, Penn.
Attn: Prof. G.L. Thompson

The George Washington Univ.
Dept. of Statistics
Washington, D.C.
Attn: Prof. Harold Bright

Stanford University
Dept. of Mathematics
Stanford, California
Attn: Prof. S. Karlin

Stanford University
Dept. of Statistics
Stanford, California
Attn: Prof. C. Stein

Logistics Research Project
The George Washington Univ.
707 22nd St, N.W.
Washington 7, D.C.
Attn: Dr. W. H. Marlow

Cornell University
Industrial and Engineering
Administration
Ithaca, New York
Attn: Prof. Henry P. Goode

University of Wisconsin
Dept. of Statistics
Madison, Wisconsin
Attn: Prof. G.E.P. Box

Stanford University
Dept. of Economics
Stanford, California
Attn: Prof. K. J. Arrow

Institute of Mathematical
Statistics
University of Copenhage
Copenhagen, Denmark
Attn: Prof. Anders Hald

Cowles Commission for Research
in Economics
Yale University
New Haven, Connecticut
Attn: Prof. T. C. Koopmans

Mass. Inst. of Technology
Cambridge, Massachusetts
Attn: Dr. R. A. Howard

Brown University
Dept. of Mathematics
Providence 12, Rhode Island
Attn: Prof. David Gale

The John Hopkins University
Dept. of Math. Statistics
34th and Charles Streets
Baltimore 18, Maryland
Attn: Prof. Geoffrey S. Watson

Northwestern University
Dept. of Mathematics
Evanston, Illinois
Attn: Dr. A. Charnes

Purdue University
Dept. of Economics
Lafayette, Indiana
Attn: Dr. Stanley Reiter

Princeton University
Dept. of Mathematics
Princeton, New Jersey
Attn: Prof. A. W. Tucker

University of California
Dept. of Economics
Berkeley, California
Attn: Dr. Roy Radner

Arthur D. Little, Inc.
30 Memorial Drive
Cambridge 42, Mass.
Attn: Mr. J. C. Hetrick

Case Institute of Technology
Systems Research Center
Cleveland, Ohio
Attn: Dr. Raymond Nelson,
Acting Director

Case Institute of Technology
Operations Research Group
Cleveland, Ohio
Attn: Prof. R. L. Ackoff

IBM Corporation
P.O.Box 218, Lamb Estate
Yorktown Heights, N.Y.
Attn: Dr. Ralph Gomory,
Research Center

Princeton University
Dept. of Econ. and Soc.
Princeton, New Jersey
Attn: Prof. O. Morgenstern

University of Maryland
Mathematics Department
College Park, Maryland
Attn: Prof. L. W. Cohen

University of California
Dept. of Engineering
Los Angeles 24, California
Attn: R. R. O'Neill

University of California
Institute of Engrg. Research
Berkeley 4, California
Attn: Prof. G. B. Dantzig

Applied Math. and Stat. Lab.
Dept. of Statistics
Stanford University
Stanford, California
Attn: Prof. H. Solomon

The Inst. of Mngmt. Sciences
Box 626
Ann Arbor, Michigan
Attn: Prof. R. M. Thrall

Cornell University
Dept. of Indust. and Eng. Admn.
Ithaca, New York
Attn: Dr. Donald L. Iglehart

University of Minnesota
School of Bus. Admn.
Minneapolis, Minnesota
Attn: Prof. Leonid Hurwicz,
Dept. of Economics

BASIC DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL REPORTS

Head, Logistics and Mathematical Statistics Branch Office of Naval Research Washington 25, D.C.	3 copies
Commanding Officer Office of Naval Research Branch Office Navy 100 Fleet Post Office New York, New York	2 copies
ASTIA Document Service Center Arlington Hall Station Arlington 12, Virginia	10 copies
Defense Logistics Studies Information Exchange Army Logistics Management Center Fort Lee, Virginia Attn: William B. Whichard	1 copy
Technical Information Officer Naval Research Laboratory Washington 25, D.C.	6 copies
Commanding Officer Office of Naval Research Branch Office 346 Broadway New York 13, New York Attn: J. Laderman	1 copy
Commanding Officer Office of Naval Research Branch Office 1030 East Green Street Pasadena 1, California Attn: Dr. A. R. Laufer	1 copy
Bureau of Supplies and Accounts Code OW Department of the Navy Washington 25, D.C.	1 copy
Institute for Defense Analyses Communications Research Division von Neumann Hall Princeton, New Jersey	1 copy